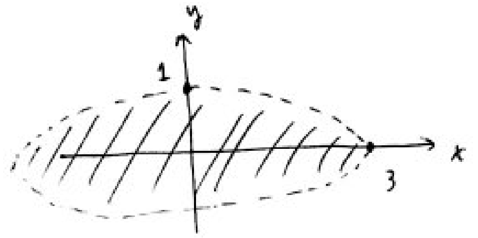
M344: Calculus III

Unit II Exam Review

1. $f(x,y) = \ln(9 - x^2 - 9y^2)$

dom: $9 - x^2 - 9y^2 > 0 \Rightarrow x^2 + 9y^2 < 9$ or $\left(\frac{x}{3}\right)^2 + y^2 < 1.$



Ellipse.

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2. $\lim_{(x,y)\to(0,0)} \frac{x^4 - 30y^2}{x^2 + 15y^2}$

Path 1: y-axis: set $x=0$. The limit becomes

$$\lim_{y\to 0} \frac{0 - 30y^2}{0 + 15y^2} = \lim_{y\to 0} \frac{-2y^2}{y^2} = -2.$$

Path 2: x-axis: set $y=0$.

$$\lim_{x\to 0} \frac{x^4 - 0}{x^2 + 0} = \lim_{x\to 0} x^2 = 0.$$

Since these limits are not equal, the limit doesnot exist.

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3. $g(t) = t + \ln t$, $f(x,y) = \frac{4 - xy}{1 + x^2y^2}$

$$g(f(x,y)) = \frac{4 - xy}{1 + x^2y^2} + \ln\left(\frac{4 - xy}{1 + x^2y^2}\right) = h(x,y).$$

This function is continuous on its domain since it is built out of a rational function and a logarithm.

Domain: $1 + x^2y^2 \neq 0 \Rightarrow (xy)^2 \neq -1$ this yields no restrictions.

$4 - xy > 0 \Rightarrow \boxed{xy < 4}$ this is the domain, hence the region where h is continuous.

4. $\frac{\partial f}{\partial x}(x_0,y_0)$ appears to be positive and

$\frac{\partial f}{\partial y}(x_0,y_0)$ appears to be negative.

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5. $$\frac{\partial f}{\partial x} = 9y\, x^{9y-1}$$

$$\frac{\partial f}{\partial y} = 9\ln(x)\, x^{9y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 9y\,(9y-1)\, x^{9y-2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (9\ln(x))^2\, x^{9y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{9}{x}x^{9y} + 9\ln(x)\,9y\, x^{9y-1} = 9x^{9y-1} + 9^2 y\ln(x)\, x^{9y-1} = (9+9^2 y\ln(x))\, x^{9y-1}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 9x^{9y-1} + 9y\cdot 9\ln(x)\, x^{9y-1} = (9+9^2 y\ln(x))\, x^{9y-1}$$

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6. $$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \quad x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2 \quad , r>0$$

so $\boxed{r = \sqrt{x^2+y^2}}$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \arctan(y/x)}.$$

Then $u(x,y) = u(r,\theta)$ and we can compute $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ in terms of r,θ.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\,\frac{-y/x^2}{1+(y/x)^2} = \frac{\partial u}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\cdot\frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial u}{\partial \theta}\cdot\frac{+y}{x^2+y^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial r}\cdot\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\cdot\frac{y}{x^2+y^2} - \frac{\partial u}{\partial \theta}\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial\theta\partial r}\frac{\partial\theta}{\partial x}\right)\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \left(\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial\theta^2}\frac{\partial\theta}{\partial x}\right)\left(\frac{y}{x^2+y^2}\right) - \frac{\partial u}{\partial\theta}\cdot\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$$

Recall:

$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan(y/x)$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{-x\frac{x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}}{x^2+y^2} = \frac{-x^2+x^2+y^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{\left(\sqrt{x^2+y^2}\right)^3}$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2yx}{\left(\sqrt{x^2+y^2}\right)^4}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}\right)\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \left(\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)\frac{-y}{\sqrt{x^2+y^2}^2} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial^2 u}{\partial \theta \partial r}\left(\frac{-y}{\sqrt{x^2+y^2}^2}\right)\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial r}\left(\frac{y^2}{\sqrt{x^2+y^2}^3}\right)$$

$$+ \left(\frac{\partial^2 u}{\partial r \partial \theta}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{-y}{\sqrt{x^2+y^2}^2}\right)\right)\left(\frac{-y}{\sqrt{x^2+y^2}^2}\right) + \frac{\partial u}{\partial \theta}\frac{2yx}{\sqrt{x^2+y^2}^4}$$

$$= \frac{\partial^2 u}{\partial r^2}\cdot\frac{x^2}{\sqrt{x^2+y^2}^2} + \frac{\partial^2 u}{\partial r \partial \theta}\left(\frac{-2xy}{\sqrt{x^2+y^2}^3}\right) + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{y^2}{\sqrt{x^2+y^2}^4}\right) + \frac{\partial u}{\partial r}\left(\frac{y^2}{\sqrt{x^2+y^2}^3}\right) + \frac{\partial u}{\partial \theta}\left(\frac{2yx}{\sqrt{x^2+y^2}^4}\right)$$

$$= \frac{r^2\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{2r^2\sin\theta\cos\theta}{r^3}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{r^2\sin^2\theta}{r^4}\frac{\partial^2 u}{\partial \theta^2} + \frac{r^2\sin^2\theta}{r^3}\frac{\partial u}{\partial r} + \frac{2r^2\sin\theta\cos\theta}{r^4}\frac{\partial u}{\partial \theta}$$

so,

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2}{r}\sin\theta\cos\theta\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2}{r^2}\sin\theta\cos\theta\frac{\partial u}{\partial \theta}}$$

Now we get to repeat all of this for $\frac{\partial^2 u}{\partial y^2}$ and add 'em up.

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}\right)$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial \theta \partial r}\frac{\partial \theta}{\partial y}\right)\left(\frac{\partial r}{\partial y}\right) + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2} + \left(\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial y}\right)\left(\frac{\partial \theta}{\partial y}\right) + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial \theta \partial r}\left(\frac{\partial \theta}{\partial y}\cdot\frac{\partial r}{\partial y}\right) + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 u}{\partial r \partial \theta}\left(\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}\right) + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial r^2}\underbrace{\left(\frac{\partial r}{\partial y}\right)^2}_{} + 2\frac{\partial^2 u}{\partial \theta \partial r}\underbrace{\left(\frac{\partial \theta}{\partial y}\frac{\partial r}{\partial y}\right)}_{} + \frac{\partial^2 u}{\partial \theta^2}\underbrace{\left(\frac{\partial \theta}{\partial y}\right)^2}_{} + \frac{\partial u}{\partial r}\underbrace{\frac{\partial^2 r}{\partial y^2}}_{} + \frac{\partial u}{\partial \theta}\underbrace{\frac{\partial^2 \theta}{\partial y^2}}_{}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{\sqrt{x^2+y^2}^3}$$

$$\frac{\partial \theta}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}^2}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{\sqrt{x^2+y^2}^4}$$

so, $$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{\sqrt{x^2+y^2}^2}\frac{\partial^2 u}{\partial r^2} + 2\frac{xy}{\sqrt{x^2+y^2}^3}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{x^2}{\sqrt{x^2+y^2}^4}\frac{\partial^2 u}{\partial \theta^2} + \frac{x^2}{\sqrt{x^2+y^2}^3}\frac{\partial u}{\partial r} - \frac{2xy}{\sqrt{x^2+y^2}^4}\frac{\partial u}{\partial \theta}$$

$$= \frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}}\frac{\partial^2 u}{\partial r^2} + \frac{2r^2\sin\theta\cos\theta}{r^3}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{r^2\cos^2\theta}{r^{4\,2}}\frac{\partial^2 u}{\partial \theta^2} + \frac{r^2\cos^2\theta}{r^3}\frac{\partial u}{\partial r} - 2\frac{r^2\cos\theta\sin\theta}{r^{4\,2}}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \cancel{\frac{2}{r}\sin\theta\cos\theta\frac{\partial^2 u}{\partial r\partial\theta}} + \frac{1}{r^2}\cos^2\theta\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\cos^2\theta\frac{\partial u}{\partial r} - \cancel{\frac{2}{r^2}\cos\theta\sin\theta\frac{\partial u}{\partial \theta}}$$

recall $$+\ \frac{\partial^2 u}{\partial x^2} = \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \cancel{\frac{2}{r}\sin\theta\cos\theta\frac{\partial^2 u}{\partial r\partial\theta}} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\sin^2\theta\frac{\partial u}{\partial r} + \cancel{\frac{2}{r^2}\cos\theta\sin\theta\frac{\partial u}{\partial \theta}}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (\sin^2\theta+\cos^2\theta)\frac{\partial^2 u}{\partial r^2} + (\sin^2\theta+\cos^2\theta)\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + (\sin^2\theta+\cos^2\theta)\frac{1}{r}\frac{\partial u}{\partial r}$$

or $$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

7. $\vec{v} = \langle 3,4 \rangle$, $\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{9+16} = 5$

so, $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \langle 3x^2+10xy, 5x^2+3y^2 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{3}{5}(3x^2+10xy) + \frac{4}{5}(5x^2+3y^2)$

$= \frac{9}{5}x^2 + 6xy + 4x^2 + \frac{12}{5}y^2$

$= \frac{29}{5}x^2 + 6xy + \frac{12}{5}y^2$

$D_{\vec{u}}[D_{\vec{u}} f] = \nabla(D_{\vec{u}} f) \cdot \vec{u} = \langle \frac{58}{5}x + 6y, 6x + \frac{24}{5}y \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$

$= \frac{3}{5}\left(\frac{58}{5}x + 6y\right) + \frac{4}{5}\left(6x + \frac{24}{5}y\right)$

$= \frac{174}{25}x + \frac{18}{5}y + \frac{24}{5}x + \frac{96}{25}y$

$= \frac{294}{25}x + \frac{186}{25}y$

$D^2_{\vec{u}} f(3,2) = \frac{294}{25}(3) + \frac{186}{25}(2) = \frac{882+372}{25} = \frac{1254}{25}$

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8. The level curve is given by $xy = 30$, or $y = \frac{30}{x}$

$y' = \frac{-30}{x^2}$ $\quad y'(6) = -\frac{30}{36} = \frac{-5}{6}$

so the tangent line is parametrized by

$\vec{r}_0 = \langle 6,5 \rangle$ and $\vec{v} = \langle 1, -5/6 \rangle$

$\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 6+t, 5 - 5t/6 \rangle}$

The gradient vector is $\nabla f = \langle y, x \rangle$ and $\nabla f(P) = \langle 5, 6 \rangle$

Then $\vec{v} \cdot \nabla f(P) = \langle 1, -5/6 \rangle \cdot \langle 5, 6 \rangle = 1(5) + (-5/6)(6) = 5 - 5 = 0$.
Therefore the gradient is orthogonal to the level curve.

9. The level curve for the constraint appears to be tangent to the level curves of the function at two points on the y-axis.

The minimum is ≈ 30 and the maximum is ≈ 59 ish.

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10. $f(x,y) = x^2+y^2+4x-4y$

$g(x,y) = x^2+y^2 \leq 81$

<u>Idea</u>: use the Second Derivative Test for any critical points in $x^2+y^2<81$ and Lagrange Multipliers for the boundary $x^2+y^2=81$

<u>Inside</u>: $\nabla f = \langle 2x+4, 2y-4\rangle = \langle 0,0\rangle$

$2x+4=0 \Rightarrow x=-2$

$2y-4=0 \Rightarrow y=2$

$(-2)^2+2^2 = 8 < 81.$

$$\hat{H}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\left.\begin{array}{l} H(f) = \det(\hat{H}(f)) = 4 > 0 \\ \frac{\partial^2 f}{\partial x^2} = 2 > 0 \end{array}\right\} \begin{array}{l} \text{so } f(-2,2) \text{ is a local minimum.} \\ f(-2,2) = (-2)^2+2^2+4(-2)-4(2) = 4+4-8-8 = -8. \end{array}$$

<u>Boundary</u>: $$\left.\begin{array}{l} \nabla f = \langle 2x+4, 2y-4\rangle \\ \nabla g = \langle 2x, 2y\rangle \end{array}\right\} \begin{array}{l} x: \ 2x+4 = 2\lambda x \\ y: \ 2y-4 = 2\lambda y \end{array} \Rightarrow \lambda = \frac{2x+4}{2x} = \frac{2y-4}{2y}$$

or $\frac{x+2}{x} = \frac{y-2}{y} \Rightarrow \cancel{xy}+2y = \cancel{xy}-2x \Rightarrow x=-y$

Then $g(x,-x) = 2x^2 = 81 \Rightarrow x = \pm\frac{9}{\sqrt{2}}$

The points are $\left(\frac{9}{\sqrt{2}}, -\frac{9}{\sqrt{2}}\right)$ and $\left(\frac{-9}{\sqrt{2}}, \frac{9}{\sqrt{2}}\right)$.

$f\left(\frac{9}{\sqrt{2}}, \frac{-9}{\sqrt{2}}\right) = \left(\frac{9}{\sqrt{2}}\right)^2 + \left(\frac{-9}{\sqrt{2}}\right)^2 + 4\left(\frac{9}{\sqrt{2}}\right) - 4\left(\frac{-9}{\sqrt{2}}\right) = \frac{81}{2}+\frac{81}{2}+\frac{36}{\sqrt{2}}+\frac{36}{\sqrt{2}} = 81+\frac{72}{\sqrt{2}}$

similarly, $f\left(\frac{-9}{\sqrt{2}}, \frac{9}{\sqrt{2}}\right) = \cdots = 81-\frac{72}{\sqrt{2}} > -8.$

So the <u>absolute max</u> is $81+\frac{72}{\sqrt{2}}$ and the <u>absolute min</u> is -8.