M344 Final Review, part II — Brief Solutions

11. Show that when Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

is written in spherical coordinates, it becomes

$$\frac{\partial^2 u}{\partial \rho^2}+\frac{2}{\rho}\frac{\partial u}{\partial \rho}+\frac{\cot\varphi}{\rho^2}\frac{\partial u}{\partial \varphi}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}=0.$$

Hint: Very carefully apply some chain rules.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right)$$

$$=\boxed{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \rho}\right)}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial \rho}\frac{\partial^2 \rho}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \varphi}\right)\frac{\partial \varphi}{\partial x}+\frac{\partial u}{\partial \varphi}\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \rho}\right)=\frac{\partial^2 u}{\partial \rho^2}\frac{\partial \rho}{\partial x}+\frac{\partial^2 u}{\partial \varphi\,\partial \rho}\frac{\partial \varphi}{\partial x}+\frac{\partial^2 u}{\partial \theta\,\partial \rho}\frac{\partial \theta}{\partial x}$$

... etc.

I'm not going to do it all, but you take all derivatives, combine like terms, then apply spherical coordinate transforms and simplify.

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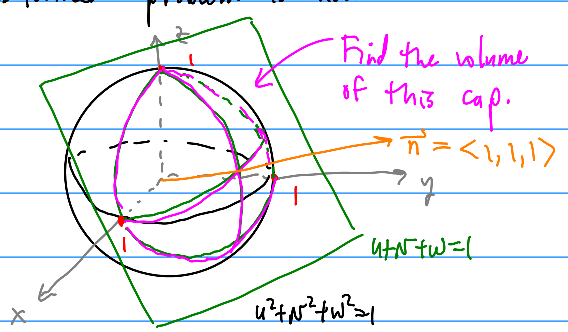
12. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $a>0$, $b>0$, $c>0$, cuts the solid ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ into two pieces. Find the volume of the smaller piece.

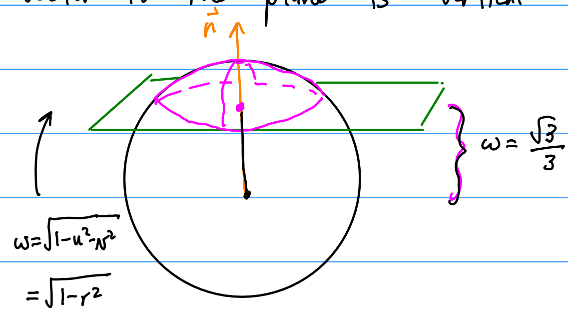
First, apply the coordinate transform $\begin{cases} u = x/a \\ v = y/b \\ w = z/c \end{cases}$ or $\begin{array}{l} x = au \\ y = bv \\ z = cw \end{array}$

The Jacobian of this transform is constant, $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc$.

The transformed problem is now:



Rotate the domain so that the normal vector to the plane is vertical



Then volume $= \int_0^{2\pi}\int_0^{1-\frac{\sqrt{3}}{3}}\int_{\frac{\sqrt{3}}{3}}^{\sqrt{1-r^2}} (abc)\, r\, dw\, dr\, d\theta$

$$= 2\pi(abc)\int_0^{1-\frac{\sqrt3}{3}} \left(-\tfrac{1}{2}\right)(-2)r\sqrt{1-r^2} - \frac{\sqrt3}{3}r \; dr$$

$$= 2\pi(abc)\left[-\frac{1}{2}\cdot\frac{2}{3}(1-r^2)^{3/2} - \frac{\sqrt3}{6}r^2\right]_0^{1-\frac{\sqrt3}{3}}$$

$$= 2\pi(abc)\left[-\tfrac{1}{3}\left(1-\left(1-\tfrac{\sqrt3}{3}\right)^2\right)^{3/2} + \frac{1}{3} - \frac{\sqrt3}{6}\left(1-\frac{\sqrt3}{3}\right)^2\right]$$

$$= 2\pi(abc)\left[-\frac{1}{3}\left(\cancel{1-1} + \frac{2\sqrt3}{3} - \frac{1}{3}\right)^{3/2} + \underline{\frac{1}{3}} - \frac{\sqrt3}{6}\left(1 - \underline{\frac{2\sqrt3}{3}} + \frac{1}{3}\right)\right]$$

$$= \boxed{2\pi(abc)\left[\frac{2}{3} - \frac{1}{3}\left(\frac{2\sqrt3-1}{3}\right)^{3/2} - \frac{2\sqrt3}{9}\right]}$$

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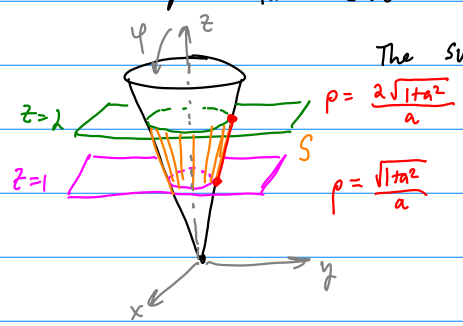
13. Find the area of the part of the cone $z^2 = a^2(x^2+y^2)$ bounded between the planes $z=1$ and $z=2$.

$z = a\sqrt{x^2+y^2}$ for $z>0$



The surface S is parametrized by $\varphi = \text{constant} = \arctan\left(\frac{1}{a}\right)$

$$z = \rho\cos\varphi = \rho\cos\left(\arctan(1/a)\right) = \frac{\rho a}{\sqrt{1+a^2}}$$

$$D\begin{cases}\rho: \frac{\sqrt{1+a^2}}{a} \to \frac{2\sqrt{1+a^2}}{a}, & \text{put } \alpha = \frac{\sqrt{1+a^2}}{a}\\ \theta: 0 \to 2\pi\end{cases}$$

So, $S: \vec r(\rho,\theta) = \left\langle \frac{\rho}{\sqrt{1+a^2}}\cos\theta, \frac{\rho}{\sqrt{1+a^2}}\sin\theta, \frac{\rho a}{\sqrt{1+a^2}}\right\rangle$

$$\left.\begin{aligned}\frac{d\vec r}{d\rho} &= \frac{1}{\sqrt{1+a^2}}\langle\cos\theta, \sin\theta, a\rangle\\ \frac{d\vec r}{d\theta} &= \frac{1}{\sqrt{1+a^2}}\langle-\rho\sin\theta, \rho\cos\theta, 0\rangle\end{aligned}\right\} \vec n = \frac{1}{1+a^2}\langle -a\rho\cos\theta, -a\rho\sin\theta, \rho\rangle$$

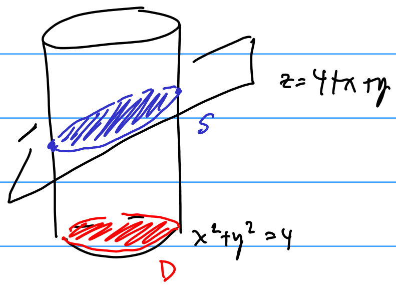
$$\|\vec n\| = \frac{1}{1+a^2}\sqrt{a^2\rho^2+\rho^2} = \frac{\rho}{\sqrt{1+a^2}}$$

$$SA = \iint_S 1\,dS = \iint_D \|\vec n\|\,dA = \int_0^{2\pi}\int_\alpha^{2\alpha}\frac{\rho}{\sqrt{1+a^2}}\,d\rho\,d\theta$$

$$= \frac{1}{\sqrt{1+a^2}}\cdot 2\pi\cdot\frac{1}{2}\rho^2\Big|_\alpha^{2\alpha}$$

$$= \frac{\pi}{\sqrt{1+a^2}}\left(4\alpha^2-\alpha^2\right) = \frac{3\pi\left(\sqrt{1+a^2}\right)^{\cancel{2}}}{a^2\cancel{\sqrt{1+a^2}}} = \boxed{\frac{3\pi\sqrt{1+a^2}}{a^2}}$$

14. Compute the surface integral $\iint_S (x^2z+y^2z)\,dS$, where S is the part of the plane $z=4+x+y$ that lies inside of the cylinder $x^2+y^2=4$.



S: $\vec r(x,y) = \langle x, y, 4+x+y\rangle$

$\vec v = \langle -1,-1,1\rangle$ since S is a plane.

$\|\vec v\| = \sqrt3$.

So the integral is: $\iint_D \left(x^2(4+x+y)+y^2(4+x+y)\right)\sqrt3\,dA$

$$=\sqrt3\iint_D (4x^2+x^3+x^2y+4y^2+xy^2+y^3)\,dA$$ Now use polar in D:

$$=\sqrt3\int_0^{2\pi}\int_0^2 \left(4r^2\cos^2\theta + r^3\cos^3\theta + r^3\cos^2\theta\sin\theta + 4r^2\sin^2\theta + r^3\cos\theta\sin^2\theta + r^3\sin^3\theta\right) r\,dr\,d\theta$$

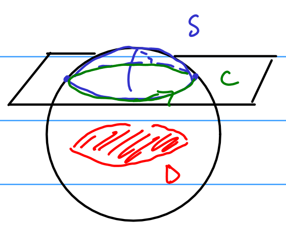
$$=\sqrt3\int_0^{2\pi}\left[4\int_0^2 r^3\,dr + \left(\cos^3\theta+\cos^2\theta\sin\theta+\cos\theta\sin^2\theta+\sin^3\theta\right)\int_0^2 r^4\,dr\right]d\theta$$

$$=32\sqrt3\,\pi + 4\sqrt3\int_0^{2\pi} \cos\theta(1-\sin^2\theta)+\cos^2\cancel{\theta\sin\theta}+\cos\theta\cancel{\sin^2\theta}+\sin\theta(1-\cancel{\cos^2\theta})\,d\theta$$

0

$$=\boxed{32\sqrt3\,\pi}$$

15. Evaluate $\iint_S \operatorname{curl}\mathbf F\cdot d\mathbf S$ where $\mathbf F=\langle x^2yz, yz^2, z^3e^{xy}\rangle$ and S is the part of the sphere $x^2+y^2+z^2=5$ that lies above the plane $z=1$. Take S to be oriented upward. [Hint: Use Stokes' Theorem.]



$z=1$ $\quad x^2+y^2+1^2=5 \quad\to\quad x^2+y^2=4$

C: $\vec r(t)=\langle 2\cos t, 2\sin t, 1\rangle$

$\vec r\,' = \langle -2\sin t, 2\cos t, 0\rangle$

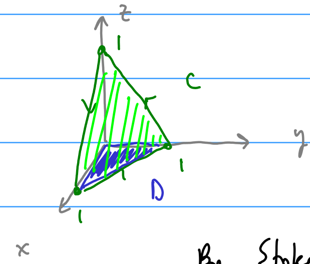
$\vec F(\vec r)=\langle 8\cos^2 t\sin t,\ 2\sin t,\ \_\_\rangle$

$\vec F\cdot d\vec r = -16\cos^2t\sin^2t + 4\cos t\sin t$ $\quad 0$

By Stokes': $\iint_S \operatorname{curl}\vec F\cdot d\vec S = \int_C \vec F\cdot d\vec r = \int_0^{2\pi} -16\left(\tfrac12\sin(2t)\right)^2 + 4\left(\tfrac12\cancel{\sin(2t)}\right)dt$

$$= -2\int_0^{2\pi} 1-\cos\underset{0}{\cancel{(4t)}}\,dt = \boxed{-4\pi}$$

16. Evaluate the path integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy, yz, zx\rangle$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented counter-clockwise when looking "from above." [Hint: Use Stokes' Theorem.]



$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ xy & yz & zx \end{vmatrix} = \langle -y, -z, -x\rangle$$

$S: x+y+z=1$ has $\vec{v} = \langle 1,1,1\rangle$

By Stokes', $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \langle -y,-z,-x\rangle\cdot\langle 1,1,1\rangle\, dA$

$$= \int_0^1\int_0^{1-y} -y-z-x\;dx\,dy = \int_0^1\int_0^{1-y} \cancel{-y}-(1-\cancel{x}-\cancel{y})\cancel{-x}\;dx\,dy$$

$$= \iint_D -1\;dA = -1\cdot \text{Area}(D) = \boxed{-\tfrac{1}{2}}$$

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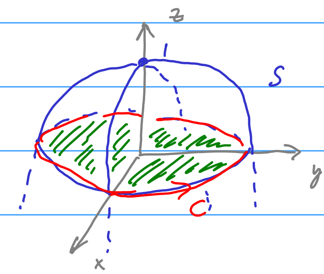
17. Find the positively oriented Jordan curve C for which the value of the path integral $\int_C (y^3-y)\,dx - 2x^3\,dy$ is a maximum.

By Green's Theorem, $\int_C (y^3-y)\,dx - 2x^3\,dy = \iint_D \underbrace{-6x^2-3y^2+1}_{f(x,y)}\;dA$



The graph of $z=f(x,y)$ is an elliptic paraboloid.

This integral will be maximized when S sits entirely above the xy-plane.

Thus C is the ellipse: $\boxed{6x^2+3y^2=1}$

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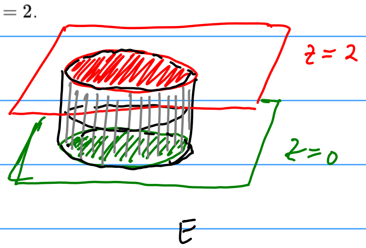
18. Use the Divergence Theorem to calculate the flux of $\mathbf{F}$ across the surface S, where $\mathbf{F} = \langle x^3, y^3, z^3\rangle$ and S is the surface bounded by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=2$.



$$\text{div}(\vec{F}) = 3x^2+3y^2+3z^2 = 3(x^2+y^2+z^2)$$

$$\text{Flux} = \iiint_E 3(x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^2 3(r^2+z^2)\,r\,dz\,dr\,d\theta$$

$$= 6\pi\int_0^1 r^3 z + \tfrac{1}{3}rz^3\Big|_{z=0}^{2}\,dr$$

$$= 6\pi\int_0^1 2r^3 + \tfrac{8}{3}r\;dr$$

$$= 6\pi\cdot\left(\tfrac{1}{2}r^4 + \tfrac{4}{3}r^2\Big|_0^1\right)$$

$$= 6\pi\cdot\left(\tfrac{1}{2}+\tfrac{4}{3}\right) = \boxed{11\pi}$$

19. Show that $\mathbf{F}$ is a conservative vector field and use this fact to evaluate the path integral, $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\begin{cases} \mathbf{F} = \overset{P}{(4x^3y^2 - 2xy^3)}\mathbf{i} + \overset{Q}{(2x^4y - 3x^2y^2 + 4y^3)}\mathbf{j}, \\ C : \mathbf{r}(t) = (t+\sin(\pi t))\mathbf{i} + (2t+\cos(\pi t))\mathbf{j}, \quad 0 \le t \le 1. \end{cases}$$

$$\left.\begin{array}{l} \frac{\partial Q}{\partial x} = 8x^3y - 6xy^2 \\ \frac{\partial P}{\partial y} = 8x^3y - 6xy^2 \end{array}\right\} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \qquad \text{conservative!}$$

$$\left.\begin{array}{l} f = \int P\,dx = \int (4x^3y^2 - 2xy^3)\,dx = x^4y^2 - x^2y^3 + C_1(y) \\ f = \int Q\,dy = \int (2x^4y - 3x^2y^2 + 4y^3)\,dy = x^4y^2 - x^2y^3 + y^4 + C_2(x) \end{array}\right\} \quad \begin{array}{l} \underline{\text{Potential function:}} \\ f(x,y) = x^4y^2 - x^2y^3 + y^4 \end{array}$$

$A = \vec{r}(0) = \langle 0, 1\rangle \qquad B = \vec{r}(1) = \langle 1, 1\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A) = (1-1+1) - (0-0+1) = 1-1 = 0.$$

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20. Suppose f is a harmonic function on an open domain $D \subseteq \mathbb{R}^2$; that is, $\Delta f = 0$ on D. Show that $\int_C \partial_y f\,dx - \partial_x f\,dy$ is independent of path in D.

Green's Theorem: $\int_C \partial_y f\,dx - \partial_x f\,dy = \iint_D -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\,dA = -\iint_D \Delta f\,dA = -\iint_D 0\,dA = 0.$