Name: Key

M344: Calculus III (Su.19)

Final Exam, part I

Thursday, 25 July 2019

WICHITA STATE UNIVERSITY

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use two 3×5 in^2 index cards of your own hand-written notes, but you may not use any electronic devices.

Each question is worth 20 points.

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1. Use a $\delta - \varepsilon$ argument to prove that the limit exists. Show enough work.

$$\lim_{(x,y)\to(0,0)} \frac{9x^2y}{x^2+y^2}$$

Assuming the limit exists it must be

$$L = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

Let $\varepsilon > 0$, and put $\delta(\varepsilon) = \underline{\varepsilon/9}$.

If $0 < \sqrt{x^2+y^2} < \delta$, then

$$\left|\frac{9x^2y}{x^2+y^2} - 0\right| = \frac{9x^2|y|}{x^2+y^2} \leq \frac{9(x^2+y^2)|y|}{x^2+y^2} = 9|y| = 9\sqrt{y^2} \leq 9\sqrt{x^2+y^2} < 9\delta = 9 \cdot \frac{\varepsilon}{9} = \varepsilon.$$

Thus, by the definition of limit,

$|f(x,y) - 0| < \varepsilon$ whenever $0 < \|\vec{x} - \vec{0}\| < \delta$ implies

$$\lim_{\vec{x}\to\vec{0}} \frac{9x^2y}{x^2+y^2} = 0. \blacksquare$$

2. Consider the space curve C parametrized by the vector function

$$\mathbf{r}(t) = \langle t^2 + 2t,\, 2t - 1,\, t^2 - 2t + 1 \rangle.$$

Find formulas for the unit tangent and unit normal vector fields along C: $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

$$\vec{r}\,'(t) = \langle 2t+2,\, 2,\, 2t-2 \rangle$$

$$\|\vec{r}\,'\| = \sqrt{(2t+2)^2 + 2^2 + (2t-2)^2} = 2\sqrt{t^2 + \cancel{2t} + 1 + 1 + t^2 - \cancel{2t} + 1} = 2\sqrt{2t^2+3}$$

and

$$\boxed{\vec{T}(t) = \left\langle \frac{t+1}{\sqrt{2t^2+3}},\, \frac{1}{\sqrt{2t^2+3}},\, \frac{t-1}{\sqrt{2t^2+3}} \right\rangle}$$

$$\vec{T}\,' = \frac{1}{2t^2+3}\left\langle \sqrt{2t^2+3}\cdot 1 - \frac{(t+1)(\cancelto{2}{4}t)}{\cancel{2}\sqrt{2t^2+3}},\, \frac{-\cancelto{2}{4}t}{\cancel{2}\sqrt{2t^2+3}},\, \sqrt{2t^2+3}\cdot 1 - \frac{(t-1)(\cancelto{2}{4}t)}{\cancel{2}\sqrt{2t^2+3}} \right\rangle$$

$$= \frac{1}{(2t^2+3)^{3/2}}\left\langle \cancel{2t^2} + 3 - \cancel{2t^2} - 2t,\, -2t,\, \cancel{2t^2} + 3 - \cancel{2t^2} + 2t \right\rangle \rightsquigarrow \langle 3-2t,\, -2t,\, 3+2t \rangle = \vec{T}\,'$$

$$\|\vec{T}\,'\| = \sqrt{(3-2t)^2 + (2t)^2 + (3+2t)^2} = \sqrt{9 - \cancel{12t} + 4t^2 + 4t^2 + 9 + \cancel{12t} + 4t^2} = \sqrt{18 + 12t^2} = \sqrt{6}\sqrt{3+2t^2}$$

and

$$\boxed{\vec{N}(t) = \left\langle \frac{3-2t}{\sqrt{6}\sqrt{3+2t^2}},\, \frac{-2t}{\sqrt{6}\sqrt{3+2t^2}},\, \frac{3+2t}{\sqrt{6}\sqrt{3+2t^2}} \right\rangle}$$

3. Find the maximum and minimum values of the function

$$f(x, y) = x^2 - y^2$$

subject to the constraint $x^2 + 4y^2 = 16$.

$$\left.\begin{array}{l} \nabla f = \langle 2x, -2y \rangle \\ \nabla g = \langle 2x, 8y \rangle \end{array}\right\} \quad \begin{array}{l} 2x = 2\lambda x \\ -2y = 8\lambda y \\ x^2 + 4y^2 = 16 \end{array} \quad \longrightarrow \quad \begin{array}{l} \lambda = 1, x \neq 0. \\ -2y = 8y \\ y = 0 \\ x = \pm 4 \end{array} \quad \longrightarrow \quad x = 0 \Rightarrow y = \pm 2$$

Plug in the points $(4,0), (-4,0), (0,2), (0,-2)$

$$\boxed{\begin{array}{ll} f(\pm 4, 0) = 16 & \leftarrow \text{Abs. Max} \\ f(0, \pm 2) = -4 & \leftarrow \text{Abs. Min} \end{array}}$$

4. Reparametrize the curve with respect to arc length measured from the point $P(0, \pi, 1)$ in the direction of increasing t.

$$\mathbf{r}(t) = \left\langle \sin(4t),\ t,\ \cos(4t) \right\rangle$$

$\dot{\vec{r}}(t) = \langle 4\cos(4t), 1, -4\sin(4t) \rangle$

$\|\dot{\vec{r}}\| = \sqrt{16\cos^2(4t) + 16\sin^2(4t) + 1} = \sqrt{17}$

$s = \int_{\pi}^{t} \sqrt{17}\, du = \sqrt{17}\,t - \sqrt{17}\,\pi$

$t = \frac{s}{\sqrt{17}} + \pi$

so

$$\boxed{\vec{r}(s) = \left\langle \sin\left(\frac{4s}{\sqrt{17}}\right), \frac{s}{\sqrt{17}} + \pi, \cos\left(\frac{4s}{\sqrt{17}}\right) \right\rangle}$$

5. Compute the directional derivative $D_{\mathbf{v}} f(P)$ where $f(x, y) = e^x \sin y - \frac{1}{2} e^x \cos y$, P is the point $(0, \pi)$, and $\mathbf{v}$ makes an angle of $\frac{\pi}{4}$ with the positive x-axis.

$$\vec{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

$$\nabla f = \left\langle e^x \sin y - \frac{1}{2} e^x \cos y,\ e^x \cos y + \frac{1}{2} e^x \sin y \right\rangle$$

$$\nabla f(P) = \left\langle \frac{1}{2}, -1 \right\rangle$$

So $$D_{\vec{v}} f(P) = \nabla f(P) \cdot \vec{v} = \frac{\sqrt{2}}{2} \langle 1, 1 \rangle \cdot \left\langle \frac{1}{2}, -1 \right\rangle = \frac{\sqrt{2}}{2} \left(\frac{1}{2} - 1 \right) = \boxed{-\frac{\sqrt{2}}{4}}$$

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