Name: Key

M344: Calculus III (Su.19)
Final Exam, part II
Friday, 26 July 2019

Wichita State University

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use two 3×5 in^2 index cards of your own hand-written notes, but you may not use any electronic devices.

Each question is worth 20 points.

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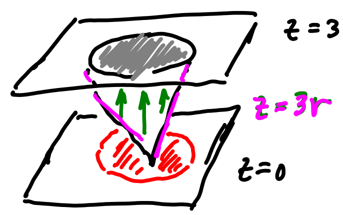
1. Use a double or triple integral to compute the volume of the part of the cone

$$z^2 = 9(x^2 + y^2)$$

that is bounded between the planes $z = 0$ and $z = 3$.



$$V = \int_0^{2\pi} \int_0^1 \int_{3r}^3 r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 (3-3r) r \, dr$$

$$= 2\pi \cdot \left(\tfrac{3}{2} r^2 - r^3 \Big|_0^1 \right)$$

$$= 2\pi \left(\tfrac{3}{2} - 1 \right) = 2\pi \cdot \tfrac{1}{2} = \boxed{\pi}$$

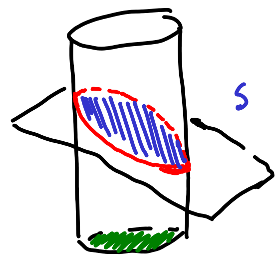
2. Let C be the curve of intersection of the plane

$$z - 2x - 3y = 0 \longrightarrow z = 2x + 3y$$

and the cylinder

$$x^2 + y^2 = 9,$$

and let $\mathbf{F}(x, y, z) = \langle z, y, x \rangle$. Use your favorite method to compute the path integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.



Stokes': $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot d\vec{S}$

where $\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ z & y & x \end{vmatrix} = \langle 0, 1-1, 0 \rangle = \langle 0, 0, 0 \rangle$

So $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \iint_S \vec{0} \cdot d\vec{S} = \boxed{0.}$

3. Consider the vector field $\mathbf{F}(x,y,z) = \langle x, y, z\rangle$. Use your favorite method to compute the flux, $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the surface of the solid bounded by the upper half sphere

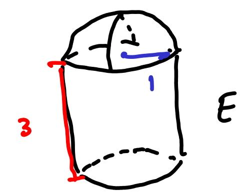
$$z = 3 + \sqrt{1 - x^2 - y^2},$$

the cylinder

$$x^2 + y^2 = 1,$$

and the disk

$$x^2 + y^2 \le 1,\ z = 0.$$



Divergence Theorem:

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\operatorname{div}\vec{F})\,dV \qquad \operatorname{div}\vec{F} = 1+1+1 = 3$$

$$= 3\iiint_E dV = 3\cdot \text{vol}(E).$$

$$= 3\left(\pi r^2 h + \frac{1}{2}\frac{4}{3}\pi r^3\right)$$

$$= 3\left(3\pi + \frac{1}{2}\frac{4}{3}\pi\right)$$

$$= \boxed{11\pi}$$

4. Show that **F** is a conservative vector field and use this fact to evaluate the path integral, $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\begin{cases} \mathbf{F} = (4x^3y^2 - 2xy^3)\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\mathbf{j}, \\ C : \mathbf{r}(t) = (t+\sin(\pi t))\mathbf{i} + (2t+\cos(\pi t))\mathbf{j}, \quad 0 \le t \le 1. \end{cases}$$

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= 8x^3y - 6xy^2 \\ \frac{\partial P}{\partial y} &= 8x^3y - 6xy^2 \end{aligned}\right\} = \text{conservative!}$$

$$f = \int P\,dx = \int 4x^3y^2 - 2xy^3\,dx = x^4y^2 - x^2y^3 \quad + C_1(y)$$

$$f = \int Q\,dy = \int 2x^4y - 3x^2y^2 + 4y^3\,dy = x^4y^2 - x^2y^3 + y^4 \quad + C_2(x)$$

The potential function is $f(x,y) = x^4y^2 - x^2y^3 + y^4$

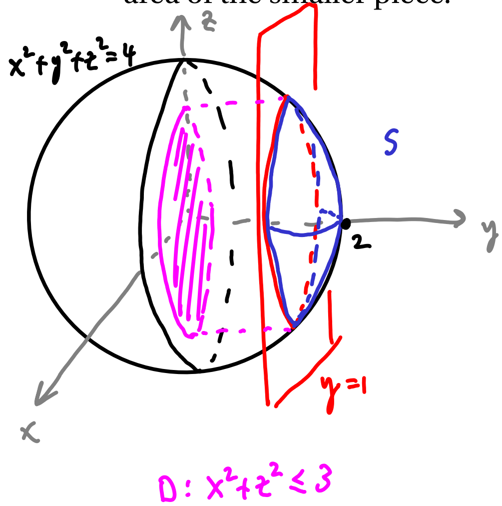
$B = \vec{r}(1) = \langle 1, 1 \rangle$

$A = \vec{r}(0) = \langle 0, 1 \rangle$

So $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$

$= 1 - 1 + 1 - (0 - 0 + 1) = 2 - 2 = 0$

5. The plane $y = 1$ cuts the sphere $x^2 + y^2 + z^2 = 4$ into two pieces. Compute the surface area of the smaller piece.



$$S: \quad \vec{r}(x,z) = \langle x, \sqrt{4-x^2-z^2}, z \rangle$$

$$\vec{r}_x = \left\langle 1, \frac{-\cancel{2}x}{\cancel{2}\sqrt{4-x^2-z^2}}, 0 \right\rangle$$

$$\vec{r}_z = \left\langle 0, \frac{-z}{\sqrt{4-x^2-z^2}}, 1 \right\rangle$$

$$\vec{v} = \vec{r}_x \times \vec{r}_z = \left\langle \frac{-x}{\sqrt{4-x^2-z^2}}, -1, \frac{-z}{\sqrt{4-x^2-z^2}} \right\rangle$$

$$\|\vec{v}\| = \frac{1}{\sqrt{4-x^2-z^2}} \sqrt{x^2 + 4 - x^2 - z^2 + z^2} = \frac{2}{\sqrt{4-x^2-z^2}}$$

$$SA = \iint_S 1\, dS = \iint_D \|\vec{v}\|\, dA = \int_0^{2\pi} - \int_0^{\sqrt{3}} \frac{-2}{\sqrt{4-r^2}} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^2 u^{-1/2}\, du\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot 2\sqrt{u}\Big|_1^2$$

$$= 2\pi \cdot (2\sqrt{2} - 2)$$

$$= \boxed{4\pi(\sqrt{2}-1)}$$

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