Name: Key

M344: Calculus III (Su.19)
Midterm Exam: Review Guide
Chapters 13 and 14

Instructions. Complete all problems on this paper, showing enough work. You may use any resources that you'd like to complete this review guide. Keep in mind that the midterm exam will be closed notes.

This assignment is optional. If submitted, it will be counted as extra credit toward your final grade.

1–4. True/False [1 point each] Write a **T** on the line if the statement is always true, and **F** otherwise. If you determine that the statement is false, you must give justification in the space provided to receive credit.

F **1.** Let $\mathbf{r}$ be a smooth vector function. If $\|\mathbf{r}(t)\| = 1$ for all t, then $\|\dot{\mathbf{r}}(t)\|$ is constant.

$\|\dot{\vec{r}}(t)\| = \frac{ds}{dt}$ is unrelated to $\|\vec{r}\|$.

T **2.** Let $\mathbf{r}$ be a smooth vector function. If $\|\mathbf{r}(t)\| = 1$ for all t, then $\dot{\mathbf{r}}(t)$ is orthgonal to $\mathbf{r}(t)$ for all t.

Proof done in class. (make sure you can prove it.)

F **3.** Let $\mathbf{r}$ be a vector function such that $\ddot{\mathbf{r}}(t)$ exists for all t, $\dot{\mathbf{r}} \neq \mathbf{0}$, and $\ddot{\mathbf{r}} \neq \mathbf{0}$. Then $\mathbf{N} = \kappa\mathbf{T}$.

$\frac{d\vec{T}}{ds} = \kappa\vec{N}$: $\kappa = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$, $\vec{N} = \frac{\dot{\vec{T}}}{\|\dot{\vec{T}}\|} \Rightarrow \kappa\vec{N} = \frac{\dot{\vec{T}}}{\|\dot{\vec{r}}\|} = \frac{d\vec{T}/dt}{ds/dt} = \frac{d\vec{T}}{ds}$

F **4.** Let f be a function of (x, y). If f has a local minimum at (a, b), then $\nabla f(a, b) = \mathbf{0}$.

$\nabla f(a,b)$ could be undefined.

e.g., $f(x,y) = \sqrt{x^2+y^2}$ at $(0,0)$

5. [1 point] Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the paraboloid $z = 2x^2 + 2y^2$.

Cylinder: $x^2+y^2=16$: $\begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases}$

paraboloid: $z = 2x^2 + 2y^2 = 2\left(4^2\cos^2 t + 4^2\sin^2 t\right) = 32\left(\cos^2 t + \sin^2 t\right) = 32$

so, $\boxed{\vec{r}(t) = \langle 4\cos t, 4\sin t, 32\rangle}$

6. [1 point] Find the point on the curve $y = e^{2x}$ at which the curvature is maximized.

$\kappa(x) = \dfrac{|y''|}{\sqrt{1+(y')^2}^{\,3}} = \dfrac{4e^{2x}}{\sqrt{1+4e^{4x}}^{\,3}}$

$y' = 2e^{2x}$

$y'' = 4e^{2x}$

$$\kappa'(x) = \frac{\sqrt{1+4e^{4x}}^{\,3} \cdot 8e^{2x} - 4e^{2x} \cdot \frac{3}{2}\sqrt{1+4e^{4x}} \cdot 16e^{4x}}{(1+4e^{4x})^3}$$

$$= \frac{\sqrt{1+4e^{4x}}\left[(1+4e^{4x})\,8e^{2x} - 6\cdot 16\, e^{6x}\right]}{(1+4e^{4x})^3}$$

This will equal 0 only if the numerator is 0:

$8e^{2x} + 32e^{6x} - 96e^{6x} = 8e^{2x}(1-8e^{4x}) = 0$

implies: $e^{4x} = \frac{1}{8} \rightarrow 4x = -\ln 8 \rightarrow x = -\frac{1}{4}\ln 8.$

Then $y(-\frac{1}{4}\ln 8) = e^{(-\frac{1}{4}\ln 8)\cdot 2} = e^{\ln 8^{-1/2}} = \frac{1}{2\sqrt{2}}$

So the point on the curve of maximum curvature is:

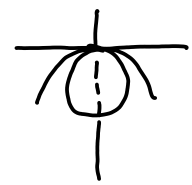
$\boxed{P\left(-\frac{1}{4}\ln 8, \frac{1}{2\sqrt{2}}\right)}$

7. [3 points] Find equations of the osculating circles to the curve $y = x^4 - x^2$ at each critical point.

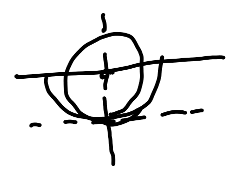
$$\left.\begin{aligned} y &= x^4 - x^2 \\ y' &= 4x^3 - 2x \\ y'' &= 12x^2 - 2 \end{aligned}\right\} \quad \kappa(x) = \frac{|12x^2-2|}{\sqrt{1+(4x^3-2x)^2}^{\,3}}$$

The critical points are: $y' = 0$: $2x(2x^2-1) = 0$ $\quad \left[\begin{aligned} x &= 0 \\ x &= \pm \frac{1}{\sqrt{2}} \end{aligned}\right.$

$\underline{x=0}$:
$$\left.\begin{aligned} \kappa(0) &= 2 \\ y''(0) &= -2 < 0 \\ y(0) &= 0 \end{aligned}\right\} \text{ so the circle is: } \boxed{x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}}$$



$\underline{x = \frac{1}{\sqrt{2}}}$:
$$\left.\begin{aligned} \kappa(1/\sqrt{2}) &= |12(1/2) - 2| = 4 \\ y''(1/\sqrt{2}) &= 4 > 0 \\ y(1/\sqrt{2}) &= 1/4 - 1/2 = -1/4 \end{aligned}\right\} \text{ the circle is: } \boxed{(x - 1/\sqrt{2})^2 + y^2 = 1/16}$$



$\underline{x = \frac{-1}{\sqrt{2}}}$:
$$\left.\begin{aligned} \kappa(-1/\sqrt{2}) &= |12(1/2) - 2| = 4 \\ y''(-1/\sqrt{2}) &= 4 > 0 \\ y(-1/\sqrt{2}) &= 1/4 - 1/2 = -1/4 \end{aligned}\right\} \text{ the circle is: } \boxed{(x + 1/\sqrt{2})^2 + y^2 = 1/16}$$

8. [1 point] Reparametrize the curve $\mathbf{r}(t) = e^t\,\mathbf{i} + e^t \sin t\,\mathbf{j} + e^t \cos t\,\mathbf{k}$ with respect to arclength from the point $(1, 0, 1)$ in the direction of increasing t.

$$\dot{\vec{r}} = \langle e^t,\ e^t \sin t + e^t \cos t,\ e^t \cos t - e^t \sin t \rangle$$

$$\|\dot{\vec{r}}\|^2 = e^{2t} + e^{2t}\left(\sin^2 t \cancel{+ 2\sin t\cos t} + \cos^2 t\right) + e^{2t}\left(\cos^2 t \cancel{- 2\cos t\sin t} + \sin^2 t\right)$$

$$= e^{2t}\left(1 + 1 + 1\right) = 3e^{2t}$$

so $\|\dot{\vec{r}}\| = \sqrt{3}\, e^t$

and $s = \int_0^t \sqrt{3}\, e^u\, du = \sqrt{3}e^t - \sqrt{3} \quad \to \quad \dfrac{s + \sqrt{3}}{\sqrt{3}} = e^t \quad \to \quad t = \ln\left|\frac{\sqrt{3}}{3}s + 1\right|$

$$\boxed{\vec{r}(s) = \left\langle \left(\tfrac{\sqrt{3}}{3}s+1\right),\ \left(\tfrac{\sqrt{3}}{3}s+1\right)\sin\left(\ln\left(\tfrac{\sqrt{3}}{3}s+1\right)\right),\ \left(\tfrac{\sqrt{3}}{3}s+1\right)\cos\left(\ln\left(\tfrac{\sqrt{3}}{3}s+1\right)\right) \right\rangle}$$

9. [1 point] Use your answer to problem **8** to compute the curvature of the space curve as a function of arclength.

$$\kappa(t) = \frac{\|\dot{\vec{T}}\|}{\|\dot{\vec{r}}\|}$$

$$\vec{r} = \langle e^t, e^t \sin t, e^t \cos t \rangle$$

$$\dot{\vec{r}} = \langle e^t,\ e^t \sin t + e^t \cos t,\ e^t \cos t - e^t \sin t \rangle$$

$$\|\dot{\vec{r}}\| = \sqrt{3}\, e^t$$

so $\vec{T} = \left\langle 1/\sqrt{3},\ 1/\sqrt{3}(\sin t + \cos t),\ \frac{1}{\sqrt{3}}(\cos t - \sin t) \right\rangle$

and $\dot{\vec{T}} = \left\langle 0,\ \frac{1}{\sqrt{3}}(\cos t - \sin t),\ \frac{1}{\sqrt{3}}(-\sin t - \cos t) \right\rangle$

and $\|\dot{\vec{T}}\| = \frac{1}{\sqrt{3}}\sqrt{\cos^2 t + \sin^2 t \cancel{- 2\cos t\sin t} + \sin^2 t + \cos^2 t \cancel{+ 2\sin t\cos t}} = \frac{\sqrt{2}}{\sqrt{3}}$

and $\kappa(t) = \dfrac{\sqrt{2}/\sqrt{3}}{\sqrt{3}\, e^t} = \dfrac{\sqrt{2}}{3e^t} = \dfrac{\sqrt{2}}{3\left(\frac{\sqrt{3}}{3}s+1\right)} = \boxed{\dfrac{\sqrt{2}}{\sqrt{3}s+3} = \kappa(s)}$

10. [3 points] Find the unit tangent, normal, and binormal vectors to the curve $\mathbf{r}(t) = \langle \sin^3 t, \cos^3 t, \sin^2 t \rangle$ at the point $t = \pi/4$.

$$\vec{r}\,'(t) = \langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t \rangle$$

$$= \sin t\cos t\,\langle 3\sin t, -3\cos t, 2 \rangle$$

$$\|\vec{r}\,'\| = \sin t\cos t\sqrt{9\sin^2 t + 9\cos^2 t + 4} = \sqrt{13}\,\sin t\cos t$$

so $\boxed{\vec{T}(t) = \left\langle \frac{3}{\sqrt{13}}\sin t, \frac{-3}{\sqrt{13}}\cos t, \frac{2}{\sqrt{13}} \right\rangle}$

$$\vec{T}\,' = \left\langle \frac{3}{\sqrt{13}}\cos t, \frac{3}{\sqrt{13}}\sin t, 0 \right\rangle$$

$$\|\vec{T}\,'\| = \frac{3}{\sqrt{13}}$$

so $\boxed{\vec{N}(t) = \langle \cos t, \sin t, 0 \rangle}$

$$\boxed{\begin{aligned} &\vec{T}(\pi/4) = \left\langle \frac{3}{\sqrt{26}}, \frac{-3}{\sqrt{26}}, \frac{2\sqrt{2}}{\sqrt{26}} \right\rangle \\ &\vec{N}(\pi/4) = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle \\ &\vec{B}(\pi/4) = \frac{1}{2\sqrt{13}}\langle -2\sqrt{2}, 2\sqrt{2}, 6 \rangle = \left\langle \frac{-2}{\sqrt{26}}, \frac{2}{\sqrt{26}}, \frac{3\sqrt{2}}{\sqrt{26}} \right\rangle \end{aligned}}$$

11. [1 point] Consider the function $f(x,y) = x^2 - 2x + y^2 + y$, and let C denote the level curve $f(x,y) = 5$. Show that the gradient vector $\nabla f(-1,1)$ is orthogonal to C at the point $(-1,1)$.

$\nabla f = \langle 2x-2, 2y+1 \rangle$

$\nabla f(-1,1) = \langle -4, 3 \rangle$

$C: \ x^2 - 2x + y^2 + y = 5$

$(x-1)^2 + (y+\frac{1}{2})^2 = 5 - 1 - \frac{1}{4}$

$(x-1)^2 + (y+\frac{1}{2})^2 = \frac{15}{4}$

$\frac{d}{dx}\left[(x-1)^2 + (y+\frac{1}{2})^2 = \frac{15}{4} \right]$

$2(x-1) + 2(y+\frac{1}{2})\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-(x-1)}{y+\frac{1}{2}}$ at $(-1,1)$: $\frac{dy}{dx} = \frac{-(-1-1)}{1+\frac{1}{2}} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$

So $T = \langle 3, 4 \rangle$

and $\nabla f(-1,1) \cdot T = \langle -4, 3 \rangle \cdot \langle 3, 4 \rangle = -12 + 12 = 0.$ $\blacksquare$

12. [1 point] Let $z = \sin(x + \sin t)$. Show that $\dfrac{\partial z}{\partial x}\dfrac{\partial^2 z}{\partial x \partial t} = \dfrac{\partial z}{\partial t}\dfrac{\partial^2 z}{\partial x^2}$.

$\frac{\partial z}{\partial x} = \cos(x + \sin t)$

$\frac{\partial^2 z}{\partial x^2} = -\sin(x + \sin t)$

$\frac{\partial z}{\partial t} = \cos(x + \sin t)\cos t$

$\frac{\partial^2 z}{\partial x \partial t} = -\sin(x+\sin t)\cos t$

$$\left.\begin{array}{l} \frac{\partial z}{\partial x} \cdot \frac{\partial^2 z}{\partial x \partial t} = -\cos(x+\sin t)\sin(x+\sin t)\cos t \\ \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2} = -\cos(x+\sin t)\sin(x+\sin t)\cos t \end{array}\right\} =$$

13. [2 points] Find the absolute maximum and minimum values of the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ on the domain $D: \{(x,y) \mid x^2+y^2 \le 4\}$.

Inside:

$$\partial_x f = (x^2+2y^2)e^{-x^2-y^2}\cdot(-2x) + 2xe^{-x^2-y^2} = 2x(1-x^2-2y^2)e^{-x^2-y^2}$$

$$\partial_y f = (x^2+2y^2)e^{-x^2-y^2}(-2y) + 4y\,e^{-x^2-y^2} = 2y(2-x^2-2y^2)e^{-x^2-y^2}$$

$$\nabla f = 2e^{-x^2-y^2}\langle x(1-x^2-2y^2),\ y(2-x^2-2y^2)\rangle \overset{?}{=} \langle 0,0\rangle$$

$x(1-x^2-2y^2)=0$
$y(2-x^2-2y^2)=0$

$(0,0)$: $f(0,0)=0$
$(0,1)$: $f(0,1)=2e^{-1}$
$(0,-1)$: $f(0,-1)=2e^{-1}$
$(1,0)$: $f(1,0)=e^{-1}$
$(-1,0)$: $f(-1,0)=e^{-1}$

Boundary: $x^2+y^2=4$ $g(x,y)=x^2+y^2$

$\nabla g = \langle 2x, 2y\rangle$: $\nabla f = \lambda \nabla g$ $\rightarrow$ $\begin{cases} \cancel{2x}(1-x^2-2y^2)e^{-x^2-y^2} = \lambda \cancel{2x} \\ \cancel{2y}(2-x^2-2y^2)e^{-x^2-y^2} = \lambda \cancel{2y} \end{cases}$

$(1-x^2-2y^2) = \lambda e^4 = 2-x^2-2y^2$

$\Rightarrow 1-x^2-2y^2 = 2-x^2-2y^2$ ✗ Cannot happen!

Another way:

$$f(x,y) = e^{-x^2-y^2}(x^2+2y^2) = e^{-\underbrace{(x^2+y^2)}_{=4}}(\underline{x^2+y^2}+y^2) = (4+y^2)e^{-4} \text{ on the boundary}$$

(does not depend on x). $y: -2 \to 2$

$z: e^{-4}y^2 + 4e^{-4}$ has min. $z(0)=4e^{-4}$

and max. $z(2)=8e^{-4}$

$\frac{2}{e} \square \frac{8}{e^4}$

$e^3 \square 4$

$>$

Total Min/Max: Absolute Min: $\boxed{z=0}$

Absolute Max: $\boxed{z=2e^{-1}}$

14. [1 point] Use differentials or a linear approximation to estimate the number $(1.98)^3\sqrt{(3.01)^2+(3.97)^2}$.

$$f(x,y,z) = x^3\sqrt{y^2+z^2}$$

$$df = 3x^2\sqrt{y^2+z^2}\,dx + \frac{x^3y}{\sqrt{y^2+z^2}}\,dy + \frac{x^3z}{\sqrt{y^2+z^2}}\,dz$$

$$f(3,3,4) = 3^3\sqrt{3^2+4^2} = 27\cdot 5 = 135$$

$$dx = \frac{-2}{100} \qquad dy = \frac{1}{100} \qquad dz = \frac{-3}{100}$$

$$f(3+dx, 3+dy, 4+dz) \approx f(3,3,4) + dz = 135 - 3.186 = \boxed{131.814}$$

$$dz = 3\cdot 3^2\sqrt{3^2+4^2}\left(\frac{-2}{100}\right) + \frac{3^3\cdot 3}{\sqrt{3^2+4^2}}\left(\frac{1}{100}\right) + \frac{3^3\cdot 4}{\sqrt{3^2+4^2}}\left(\frac{-3}{100}\right) = \frac{-270}{100} + \frac{81}{500} - \frac{324}{500} = \frac{-2700+162-648}{1000}$$

$$= -3.186$$

15. [1 point] Let $f(x,y) = x^2e^{-y}$. Compute the second directional derivative $D^2_{\mathbf{v}}f(-2,0)$, where $\mathbf{v} = \langle -3, 4\rangle$.

$$\vec{u} = \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle$$

$$D_{\vec{u}}f = \langle 2xe^{-y}, -x^2e^{-y}\rangle \cdot \left\langle \frac{-3}{5}, \frac{4}{5}\right\rangle = \frac{-6}{5}xe^{-y} - \frac{4}{5}x^2e^{-y}$$

$$D^2_{\vec{u}}f = \left\langle -\frac{6}{5}e^{-y} - \frac{8}{5}xe^{-y}, \frac{6}{5}xe^{-y} + \frac{4}{5}x^2e^{-y}\right\rangle \cdot \left\langle -\frac{3}{5}, 4/5\right\rangle = \frac{18}{25}e^{-y} + \frac{24}{25}xe^{-y} + \frac{24}{25}xe^{-y} + \frac{16}{25}x^2e^{-y}$$

$$= \frac{1}{25}(18 + 48x + 16x^2)e^{-y}$$

$$D^2_{\vec{u}}f(-2,0) = \frac{1}{25}(18 - 96 + 64) = \boxed{\frac{-14}{25}}$$