Name: Key

M344: Calculus III (Su.19)
Midterm Exam
Chapters 13 and 14

WICHITA STATE UNIVERSITY

Instructions. Complete all problems on this paper, showing enough work. You may use one 3×5 in^2 card of your own hand-written notes. You may not use any elecronic devices.

1–5. True/False [4 points each] Write a **T** on the line if the statement is always true, and **F** otherwise. If you determine that the statement is false, you must give justification in the space provided to receive credit.

T **1.** Suppose f is a twice continuously differentiable function. The curvature is 0 at every inflection point of the graph of $y = f(x)$.

$\kappa(x) = \frac{|f''(x)|}{\sqrt{1+f'(x)^2}^3}$ and $f''(x) = 0$ at inflection points.

F **2.** Different parametrizations of the same curve result in identical tangent vectors at a given point on the curve.

different params could result in opposite signs.

F **3.** Let f be a function of (x, y); then $\lim_{(x,y)\to(2,5)} f(x, y) = f(2,5)$.

f must be continuous at $(2,5)$.

T **4.** Let $\mathbf{k} = \langle 0,0,1\rangle$, and f be a function of (x, y, z); then $D_{\mathbf{k}} f(x, y, z) = \partial_z f(x, y, z)$.

$D_{\vec{k}} f = \nabla f \cdot \vec{k} = \langle \partial_x f, \partial_y f, \partial_z f\rangle \cdot \langle 0,0,1\rangle = 0 + 0 + \partial_z f = \partial_z f.$

T **5.** Let $f(x, y) = \sin x + \sin y$. Then $|D_{\mathbf{u}} f(x, y)| \leq \sqrt{2}$ for all points (x, y) and all unit vectors $\mathbf{u}$ in $\mathbb{R}^2$.

$|D_{\vec{u}} f| \leq \|\nabla f\| = \sqrt{\partial_x f^2 + \partial_y f^2} = \sqrt{\cos^2 x + \cos^2 y} \leq \sqrt{1+1} = \sqrt{2}.$

6–13. Multiple Choice [5 points each] Write the letter corresponding to the best answer on the line provided.

__A__ **6.** Let $f(x,y) = \ln\left(\sin^2(x) + \cos^2(y)\right)$. Compute $\dfrac{\partial f}{\partial x}$.

A. $\dfrac{2\sin x \cos x}{\sin^2 x + \cos^2 y}$ **B.**

C. **D.**

__D__ **7.** Let $g(x,y,z) = \dfrac{x^2 y}{z^3}$. Compute $\dfrac{\partial g}{\partial z}$.

A. **B.**

C. **D.** $\dfrac{-3x^2 y}{z^4}$

__D__ **8.** Let $z = x^2 \sin(xy)$. Compute dz.

A. **B.**

C. **D.** $\left(2x\sin(xy) - x^2 y\cos(xy)\right)dx - x^3\cos(xy)\,dy$

__B__ **9.** Let $f(x,y) = (x-1)^2 - 3(y-2)^2$. Find the linearization of f at the point $P(3,2)$.

A. **B.** $L(x,y) = 4 + 4(x-3)$

C. **D.**

A **10.** Let $f(x,y) = (x-1)^2 - 3(y-2)^2$. Find a formula for the directional derivative, $D_{\mathbf{v}} f(x,y)$, where $\mathbf{v} = \langle -12, 5 \rangle$.

A. $-\dfrac{24}{13}(x-1) - \dfrac{30}{13}(y-2)$ **B.**

C. **D.**

C **11.** Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be a smooth vector function. Which of the following formulas is <u>not</u> a valid formula for the curvature of $\mathbf{r}$?

A. **B.**

C. $\kappa = \dfrac{|\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}|}{\|\dot{\mathbf{r}}\|^3}$ **D.**

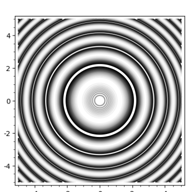
12–13. Consider the function $f(x,y) = 5 + \sqrt{9 - x^2 - y^2}$.

B **12.** What is the range of f?

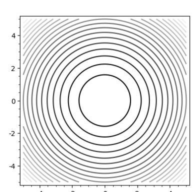
A. **B.** $[5,8]$

C. **D.**

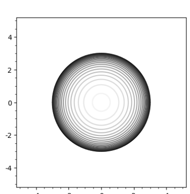
C **13.** Choose the graph that best represents the level curves of f.



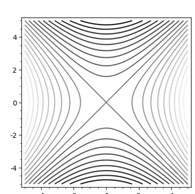
A.



B.



C.



D.

14. [30 points] Let $\mathbf{r}$ be a smooth vector function parametrizing a space curve C. Prove that the unit tangent vector field $\mathbf{T}$ on C satisfies $\mathbf{T} \perp \dot{\mathbf{T}}$ at all points along C.

On one hand, $\|\vec{T}\|=1$.

On the other hand, $\|\vec{T}\|^2 = \vec{T}\cdot\vec{T}$

So, $\vec{T}\cdot\vec{T} = 1$.

Taking the derivative implicitly,

$$\frac{d}{dt}(\vec{T}\cdot\vec{T} = 1) \Rightarrow \dot{\vec{T}}\cdot\vec{T} + \vec{T}\cdot\dot{\vec{T}} = 0 \Rightarrow 2(\vec{T}\cdot\dot{\vec{T}}) = 0$$

$$\Rightarrow \vec{T}\cdot\dot{\vec{T}} = 0 \Rightarrow \vec{T}\perp\dot{\vec{T}}. \blacksquare$$

15. [30 points] Let f be a differentiable function of two variables, and let P be a point in its domain. Prove that the directional derivative $D_{\mathbf{u}}f(P)$ is maximum when $\mathbf{u}$ is in the same direction as P, and $\max\limits_{\|\mathbf{u}\|=1} D_{\mathbf{u}}f(P) = \|\nabla f(P)\|$.

$$D_{\vec{u}}f(P) = \|\nabla f(P)\|\,\|\vec{u}\|\cos\theta = \|\nabla f(P)\|\cos\theta.$$

$\cos\theta$ is maximized when $\theta=0$, whence $\cos 0 = 1$.

So $\max\limits_{\|\vec{u}\|=1} D_{\vec{u}}f(P) = \|\nabla f(P)\|$

and this occurs when $\theta=0$, so $\vec{u}$ is in the same direction as $\nabla f(P)$. $\blacksquare$

16. [40 points] Find the maximum and minimum values of the function $f(x, y) = x^2 - y^2$ subject to the constraint $x^2+4y^2 = 16$. Leave your answers in exact form (no decimals).

$\nabla f = \langle 2x, -2y \rangle$

$g(x,y) = x^2 + 4y^2$

$\nabla g = \langle 2x, 8y \rangle$

$2x = 2\lambda x \rightarrow \lambda = 1$ if $x \neq 0$.

$8y = -2\lambda y \rightarrow 8y = -2y \Rightarrow y = 0$.

whence $x^2 + 4 \cdot 0^2 = 16 \Rightarrow x = \pm 4$.

$f(\pm 4, 0) = 16$

On the other hand $8y = -2\lambda y$ w/ $y \neq 0 \Rightarrow \lambda = -4$.

Then $2x = -8x \Rightarrow x = 0$, and $y = \pm 2$.

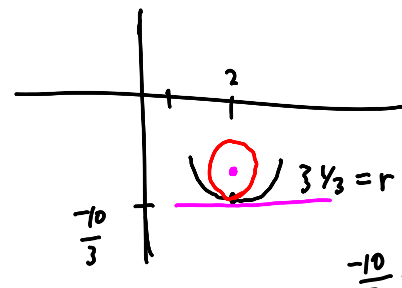
$f(0, \pm 2) = 0 - 4 = -4$

So the absolute max is 16 and

the absolute min is -4.

17. [40 points] Find an equation of the osculating circle to the curve $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x$ at $x = 2$.

$y(2) = \frac{8}{3} - 2 - 4 = \frac{8}{3} - \frac{18}{3} = \frac{-10}{3}$



$y' = x^2 - x - 2$ $\quad$ $y'' = 2x - 1$

$y'(2) = 4 - 2 - 2 = 0.$ $\quad$ $y''(2) = 4 - 1 = 3$

$\frac{-10}{3} + \frac{1}{3} = -3$

$\kappa(2) = \frac{3}{\sqrt{1+0}^{\,3}} = 3$ $\quad$ so $\quad r = 1/3$

and the circle is:

$$\boxed{(x-2)^2 + (y+3)^2 = \frac{1}{9}}$$

scratch page