Name: Key

M344: Calculus III (Su.19)

Good Problems 1

Sections 13.1–13.3

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may not use any notes or electronic devices. All you need is a pencil and your brain.

[30 pts] **1.** Consider the vector function $\mathbf{r}(t) = \left\langle \ln(2-t), \frac{t}{\sqrt{9-t^2}}, 2^t \right\rangle$.

[10] *a.*) What is the domain of $\mathbf{r}$?

$$\left.\begin{array}{l} x:\ 2-t>0 \Rightarrow t<2 \\ y:\ 9-t^2 \geq 0 \Rightarrow t^2 < 9 \Rightarrow -3 < t < 3 \\ z:\ t \in \mathbb{R}. \end{array}\right\} \quad \boxed{\operatorname{dom}(\vec{r}) = -3 < t < 2} \text{ or } \boxed{t \in (-3,2)}$$

[10] *b.*) Compute the derivative, $\dot{\mathbf{r}}(t)$.

$$\dot{\vec{r}}(t) = \left\langle \frac{-1}{2-t}, \frac{\sqrt{9-t^2} - t\frac{-2t}{2\sqrt{9-t^2}}}{9-t^2}, 2^t \ln(2) \right\rangle$$

$$\dot{\vec{r}}(t) = \left\langle \frac{-1}{2-t}, \frac{9-t^2+t^2}{\sqrt{9-t^2}^3}, 2^t \ln(2) \right\rangle = \left\langle \frac{1}{t-2}, \frac{9}{\sqrt{9-t^2}^3}, 2^t \ln(2) \right\rangle$$

[10] *c.*) Compute the antiderivative, $\int \mathbf{r}(t)\,dt$.

$$\int \vec{r}\,dt = \left\langle -(2-t)\ln(2-t) + (2-t), -\frac{2}{2}\sqrt{9-t^2}, \frac{2^t}{\ln(2)} \right\rangle + \vec{C}$$

$$= \left\langle (t-2)\ln(2-t) - (t-2), -\sqrt{9-t^2}, \frac{2^t}{\ln(2)} \right\rangle + \vec{C}$$

[15 pts] **2.** Let $\mathbf{r}$ be a smooth vector function such that $\mathbf{r}(t) \neq \mathbf{0}$. Prove that

$$\frac{d}{dt}\left[\|\mathbf{r}(t)\|\right] = \frac{\mathbf{r}(t)\cdot\dot{\mathbf{r}}(t)}{\|\mathbf{r}(t)\|}.$$

Hint: $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$.

$$\frac{d}{dt}\left[\|\vec{r}(t)\|\right] = \frac{d}{dt}\sqrt{\vec{r}\cdot\vec{r}} = \frac{1}{2\sqrt{\vec{r}\cdot\vec{r}}}\left(\vec{r}\cdot\dot{\vec{r}} + \dot{\vec{r}}\cdot\vec{r}\right)$$

$$= \frac{\cancel{2}(\vec{r}\cdot\dot{\vec{r}})}{\cancel{2}\sqrt{\vec{r}\cdot\vec{r}}}$$

$$= \frac{\vec{r}\cdot\dot{\vec{r}}}{\|\vec{r}\|} \quad \checkmark$$

[15 pts] **3.** Find $\mathbf{r}(t)$ if $\dot{\mathbf{r}}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + te^t\,\mathbf{k}$, and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\vec{r} = \int \dot{\vec{r}}\,dt = \left\langle \tfrac{1}{2}t^2 + C_1,\ e^t + C_2,\ te^t - e^t + C_3 \right\rangle$$

$$\vec{r}(0) = \left\langle C_1,\ 1 + C_2,\ -1 + C_3 \right\rangle = \langle 1, 1, 1\rangle$$

$$\Rightarrow \vec{C} = \langle 1, 0, 2\rangle$$

and

$$\boxed{\vec{r}(t) = \left\langle \tfrac{1}{2}t^2 + 1,\ e^t,\ te^t - e^t + 2 \right\rangle}$$

Consider the curve C parametrized by the vector function

$$\mathbf{r}(t) = \langle \cos t, \sin t, \ln(\cos t) \rangle.$$

[15 pts] **4.** Find the unit tangent vector field, $\mathbf{T}(t)$, along C.

$$\dot{\vec{r}} = \left\langle -\sin t, \cos t, \frac{-\sin t}{\cos t} \right\rangle = \langle -\sin t, \cos t, -\tan t \rangle$$

$$\|\dot{\vec{r}}\| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1 + \tan^2 t} = \sec t$$

$$\vec{T} = \left\langle -\frac{\sin t}{\sec t}, \frac{\cos t}{\sec t}, \frac{-\tan t}{\sec t} \right\rangle$$

or

$$\vec{T} = \langle -\sin t \cos t, \cos^2 t, -\sin t \rangle$$

[15 pts] **5.** Find the arc length of the portion of the curve on the interval $0 \le t \le \frac{\pi}{4}$.

$$s = \int_0^{\pi/4} \|\dot{\vec{r}}\| \, dt = \int_0^{\pi/4} \sec t \, dt = \ln|\sec t + \tan t| \Big|_0^{\pi/4}$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0|$$

$$= \ln(\sqrt{2} + 1).$$

[10 pts] **6.** Reparametrize the curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2+1} - 1\right)\mathbf{i} + \frac{2t}{t^2+1}\mathbf{j}$$

with respect to arc length measured from the point $(1,0)$ in the direction of increasing t. Simplify as much as possible.

Note: Grade this one generously.

$$\dot{\vec{r}} = \left\langle \frac{-2t\cdot 2}{(t^2+1)^2}, \frac{2(t^2+1) - 2t(2t)}{(t^2+1)^2} \right\rangle$$

$$= \frac{1}{(t^2+1)^2}\left\langle -4t, 2-2t^2 \right\rangle$$

$$\|\dot{\vec{r}}\| = \frac{\sqrt{16t^2 + 4 - 8t^2 + 4t^4}}{(t^2+1)^2} = \frac{\sqrt{4t^4 + 8t^2 + 4}}{(t^2+1)^2}$$

$$= \frac{2\sqrt{t^4 + 2t^2 + 1}}{(t^2+1)^2} = \frac{2\sqrt{(t^2+1)^2}}{(t^2+1)^2} = \frac{2}{(t^2+1)}$$

$$s = \int_0^t \frac{2}{u^2+1}\,du = 2\arctan t$$

so $t = \tan\left(\frac{s}{2}\right)$, and

$$\vec{r}(s) = \left\langle \frac{2}{\tan^2\left(\frac{s}{2}\right)+1} - 1, \frac{2\tan(s/2)}{\tan^2\left(\frac{s}{2}\right)+1} \right\rangle$$

$$= \left\langle 2\cos^2\left(\frac{s}{2}\right) - 1, 2\sin(s/2)\cos(s/2) \right\rangle$$

$$\boxed{\vec{r}(s) = \langle \cos(s), \sin(s) \rangle}$$