Name: Key
M344: Calculus III (Su.19)
Good Problems 2
Sections 13.3-4, 14.1

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use your own hand-written notes, but you may not use any electronic devices.

1. [10 points] Show that the curvature of a circle of radius $a > 0$ is constant: $\kappa = \frac{1}{a}$.

$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$

$\dot{\vec{r}}(t) = \langle -a\sin t, a\cos t\rangle$ $\qquad \|\dot{\vec{r}}\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$

$\ddot{\vec{r}}(t) = \langle -a\cos t, -a\sin t\rangle$

$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 0, 0, a^2\sin^2 t + a^2\cos^2 t\rangle$

$= \langle 0, 0, a^2\rangle$

$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = a^2$

So,

$$\kappa(t) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = \frac{a^2}{a^3} = \frac{1}{a}. \quad \blacksquare$$

There are at least 3 other ways to do this that are equivalent.

2. [10 points] Consider a plane curve C parametrized by a vector function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ satisfying $\dot{\mathbf{r}}(t) \neq \mathbf{0}$. Show that the curvature of C is given by

$$\kappa(t) = \frac{\left|\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)\right|}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}^3}.$$

Hint: Regard C as living in the xy-plane emebbed in $\mathbb{R}^3$.

$\vec{r} = \langle x, y\rangle$

$\dot{\vec{r}} = \langle \dot{x}, \dot{y}\rangle \to \|\dot{\vec{r}}\| = \sqrt{\dot{x}^2 + \dot{y}^2}$

$\ddot{\vec{r}} = \langle \ddot{x}, \ddot{y}\rangle$

$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 0, 0, \dot{x}\ddot{y} - \ddot{x}\dot{y}\rangle \to \|\dot{\vec{r}} \times \ddot{\vec{r}}\| = |\dot{x}\ddot{y} - \ddot{x}\dot{y}|$

$$\kappa(t) = \frac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{\sqrt{\dot{x}^2 + \dot{y}^2}^3} \quad \blacksquare$$

3. [30 points] Find an equation of the osculating circle to the curve at $x = 2$.

$$y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x$$

The point on the curve is: $(2, y(2))$ $\quad y(2) = \frac{8}{3} - 2 - 4 = \frac{8}{3} - \frac{18}{3} = \frac{-10}{3}$

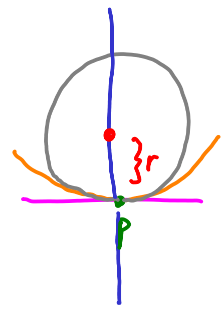
so $P = (2, \frac{-10}{3})$

The curvature is $\varkappa(x) = \frac{|f''(x)|}{\sqrt{1+f'(x)^2}^3} = \frac{|y''|}{\sqrt{1+y'^2}^3} = \frac{|3|}{\sqrt{1+0^2}^3} = 3$

$y' = x^2 - x - 2 \qquad y'(2) = 4 - 2 - 2 = 0$

$y'' = 2x - 1 \qquad y''(2) = 4 - 1 = 3$

Thus the radius of the circle is $r = \frac{1}{\varkappa} = \frac{1}{3}$.



Since $y' = 0$ and $y'' > 0$ then the curve looks like:

and the center lies directly above P.

The center is thus $(2, \frac{-10}{3} + \frac{1}{3}) = (2, -3)$

And the circle is:

$$\boxed{(x-2)^2 + (y+3)^2 = \left(\frac{1}{3}\right)^2}$$

4. [20 points] Find the tangential and normal components of the acceleration vector. Simplify as much as possible. (You do not need to find **T** and **N**).

$$\mathbf{r}(t) = \langle t, 2e^t, e^{2t} \rangle$$

Recall:

$a_T = \dot{v}$ so we need $\kappa, v,$ and $\dot{v}$ to know the acceleration components.

$a_N = \kappa v^2$

$\dot{\vec{r}} = \langle 1, 2e^t, 2e^{2t} \rangle$

$\|\dot{\vec{r}}\| = v = \sqrt{1 + 4e^{2t} + 4(e^{2t})^2} = \sqrt{(1+2e^{2t})^2} = \boxed{1 + 2e^{2t} = v(t)}$

$\boxed{\dot{v}(t) = 4e^{2t}}$

$\ddot{\vec{r}} = \langle 0, 2e^t, 4e^{2t} \rangle$

$\dot{\vec{r}} \times \ddot{\vec{r}} = \langle 4e^{3t}, -4e^{2t}, 2e^t \rangle$

$\|\dot{\vec{r}} \times \ddot{\vec{r}}\| = 2e^t \sqrt{4e^{4t} + 4e^{2t} + 1} = 2e^t \sqrt{(2e^{2t}+1)^2} = 2e^t(2e^{2t}+1)$

$\kappa(t) = \dfrac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} \Rightarrow \kappa v^2 = \dfrac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|} = \dfrac{2e^t(\cancel{2e^{2t}+1})}{\cancel{1+2e^{2t}}} = 2e^t$

So, $\boxed{a_T(t) = 4e^{2t}}$

$\boxed{a_N(t) = 2e^t}$

5. [5 points each] Match each function of 2 variables to its corresponding graph.

F i. $f(x,y) = \sin(x^2 + y^2)$

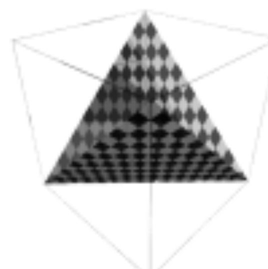
A.

B.

A ii. $f(x,y) = |x| + |y|$

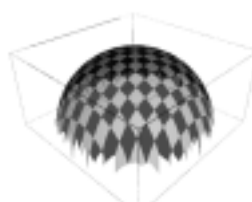
C.

D.



B iii. $f(x,y) = x^2 + y^2$

E.



F.

6. Consider the function $f(x, y) = \ln(9 - x^2 - y^2)$.

i. [10 points] Find and plot the domain of f. Be sure to properly label the graph.

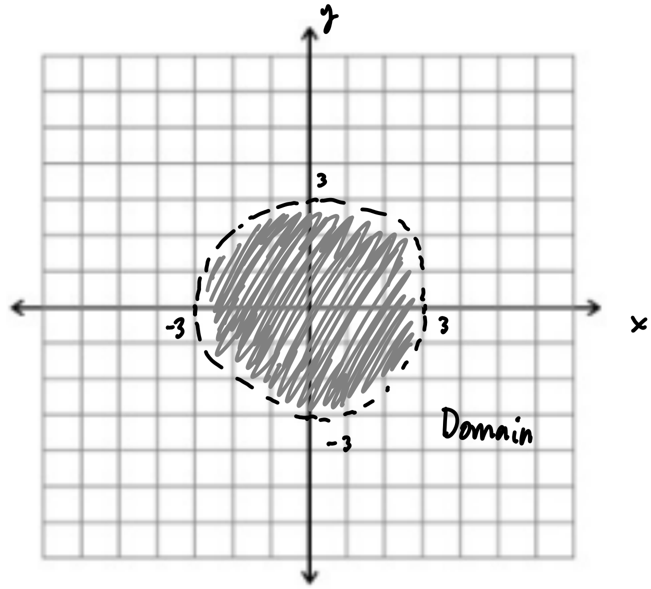
$9-x^2-y^2>0$

$x^2+y^2<9$

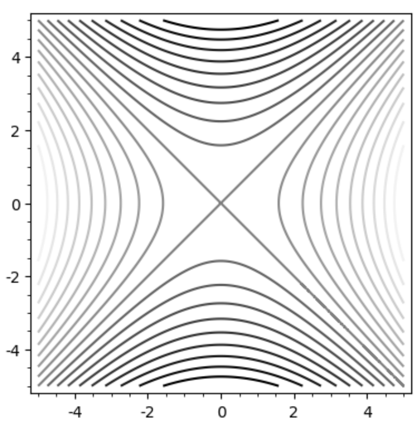
$x^2+y^2<3^2$

Inside disk of radius 3.

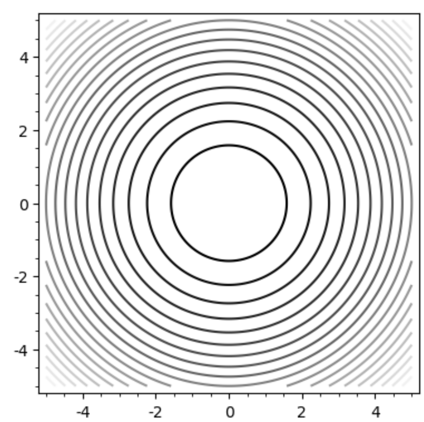
No boundary



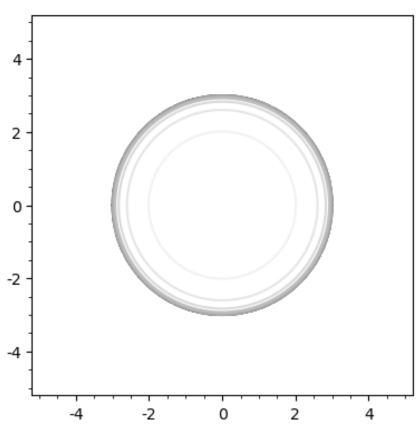
C **ii.** [5 points] Choose the plot that best represents the level curves of f.



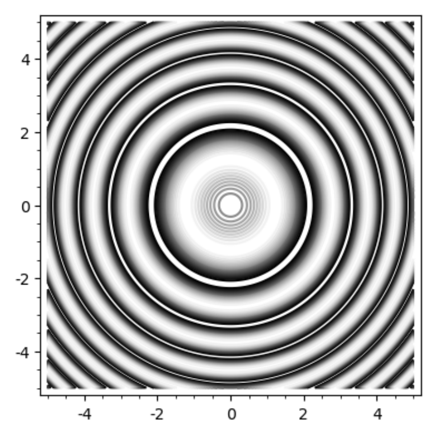
A.



B.



C.



D.

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