Name: Key

M344: Calculus III (Su.19)

Good Problems 3

Sections 14.2-14.5

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use your own hand-written notes, but you may not use any electronic devices.

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1. [10 points] Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$$

Path 1: $y=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = \lim_{x\to 0} \frac{0}{x^2} = 0$

Path 2: $x=y^4$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} = \lim_{y\to 0} \frac{y^8}{y^8+y^8} = \lim_{y\to 0} \frac{y^8}{2y^8} = \frac{1}{2}$

Since the limits along different paths do not coincide, then the limit does not exist.

2. [10 points] Use an $\varepsilon-\delta$ argument to prove that the limit does exist.

$$\lim_{(x,y)\to(0,0)} \frac{5xy^2}{x^2+y^2}$$

Along $y=0$: $\displaystyle\lim_{x\to 0} \frac{0}{x^2} = 0 = L$

Let $\varepsilon>0$ and put $\delta(\varepsilon) = \varepsilon/5$.

If $0 < \sqrt{x^2+y^2} < \delta$, then

$$|f(x,y)| = \left|\frac{5xy^2}{x^2+y^2}\right| = \frac{5|x|y^2}{x^2+y^2} \le 5|x| \le 5\sqrt{x^2+y^2} < 5\delta = 5\boxed{\frac{\varepsilon}{5}} = \varepsilon.$$

Since $\dfrac{y^2}{x^2+y^2} \le 1$ and $|x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$.

Thus, by the definition of limit, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5xy^2}{x^2+y^2} = 0.$ ∎

3. [10 points] Use polar coordinates to find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

Hint: In polar coordinates $r = \sqrt{x^2+y^2}$ and $r \to 0^+$ as $(x, y) \to (0,0)$.

$x^2+y^2=r^2$, so the limit becomes

$$\lim_{(x,y)\to(0,0)} \frac{e^{-(x^2+y^2)}-1}{x^2+y^2} = \lim_{r\to 0^+} \frac{e^{-r^2}-1}{r^2} = \frac{0}{0} \quad \text{L'Hôpital!}$$

$$= \lim_{r\to 0^+} \frac{-\cancel{2r}e^{-r^2}}{\cancel{2r}} = \lim_{r\to 0^+} -e^{-r^2} = \boxed{-1}$$

4. [15 points] Use the <u>limit definition</u> of partial derivative to compute $\partial_y f$ for the function $f(x, y) = xy^2 - x^3y$. You must use the limit definition to receive credit.

$$\partial_y f = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{x(y+h)^2 - x^3(y+h) - xy^2 + x^3y}{h}$$

$$= \lim_{h\to 0} \frac{x(y^2+2yh+h^2) - x^3y - x^3h - xy^2 + x^3y}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{xy^2} + 2xyh + xh^2 - \cancel{x^3y} - x^3h - \cancel{xy^2} + \cancel{x^3y}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2xy + xh - x^3)}{\cancel{h}}$$

$$= \boxed{2xy - x^3 = \partial_y f}$$

5. [15 points] Find an equation of the tangent plane to the surface $z = (x+2)^2 - 2(y-1)^2 - 5$ at the point $(2,3,3)$.

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= 2(x+2) \\ \frac{\partial z}{\partial y} &= -4(y-1) \end{aligned}\right\} \quad \begin{aligned} \partial_x z(2,3) &= 8 \\ \partial_y z(2,3) &= -8 \end{aligned}$$

$$T_pS: \quad z = f(2,3) + \partial_x f(2,3)(x-2) + \partial_y f(2,3)(y-3)$$

so

$$\boxed{z = 3 + 8(x-2) - 8(y-3)}$$

6. [10 points] Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

Hint: The volume of a right circular cylinder is $V(r,h) = \pi r^2 h$. Find dV.

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$dV = 2\pi r h \cdot dr + \pi r^2 \cdot dh$$

$$\left.\begin{aligned} r &= 4 \text{ cm} \\ h &= 12 \text{ cm} \\ dr = dh &= \tfrac{1}{25} \text{ cm} \end{aligned}\right\} \quad dV = \pi\left(\frac{2\cdot 4\cdot 12}{25} + \frac{4^2}{25}\right) \text{cm}^3 = \pi\left(\frac{16(7)}{25}\right) \text{cm}^3 = \boxed{\frac{112\pi}{25} \text{ cm}^3}$$

7. [10 points] Let $F = F(x, y, z)$ be a differentiable function, and suppose $z = z(x, y)$ is an implicit function defined by the formula $F(x, y, z) = k$, where k is a constant. Show that

$$\frac{\partial z}{\partial x} = \frac{-\partial_x F}{\partial_z F}.$$

Hint: Apply the chain rule to F.

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}}$$

8–11. [5 points each] Compute the indicated partial derivatives of the given functions.

C **8.** $f(x, y) = \dfrac{x}{(x + y)^2}$; Find $\partial_x f$.

$$\partial_x f = \frac{1}{(x+y)^2} - \frac{2x}{(x+y)^3}$$

$$= \frac{x+y-2x}{(x+y)^3} = \frac{y-x}{(x+y)^3}$$

A.

B.

C. $\dfrac{y - x}{(x + y)^3}$

D.

Product Rule (This can also be done by a quotient rule)

D **9.** $f(x, y) = \dfrac{x}{(x + y)^2}$; Find $\partial_y f$.

$$\partial_y f = \frac{-2x}{(x+y)^3}$$

A.

B.

C.

D. $-\dfrac{2x}{(x + y)^3}$

Chain Rule

$\underline{\quad B \quad}$ **10.** $F(x,y) = \int_x^y \cos(e^t)\,dt$; Find $\partial_x F$.

$\int_x^y \cos(e^t)dt = -\int_y^x \cos(e^t)dt$

A.

B. $-\cos(e^x)$

C.

D.

$\frac{\partial}{\partial x}\left(-\int_y^x \cos(e^t)dt\right) = -\cos(e^x)$ by FTC!

$\underline{\quad A \quad}$ **11.** $w = z^{y/x}$; Find $\partial_y w$.

$\partial_y w = z^{y/x} \cdot \ln(z) \cdot \frac{1}{x}$

Chain Rule.

A. $\frac{z^{\frac{y}{x}} \ln(z)}{x}$

B.

C.

D.

12. [Bonus: 10 points] Let $a > 0$ be a constant. Show that the function

$$u(x,t) = \sin(x - at) + \ln(x + at)$$

is a solution of the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

$$\frac{\partial u}{\partial t} = -a\cos(x-at) + \frac{a}{x+at}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = -a^2 \sin(x-at) - \frac{a^2}{(x+at)^2}}$$

$$\frac{\partial u}{\partial x} = \cos(x-at) + \frac{1}{x+at}$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x-at) - \frac{1}{(x+at)^2}$$

$$\boxed{a^2\frac{\partial^2 u}{\partial x^2} = -a^2 \sin(x-at) - \frac{a^2}{(x+at)^2}}$$

These are equal, so $u(x,t) = \sin(x-at) + \ln(x+at)$ is a solution of the PDE.

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