Name: Key
M344: Calculus III (Su.19)
Good Problems 4
Sections 15.1, 15.2, 15.9

Instructions. Complete all problems on this paper. You may use any resources that you'd like, but be sure to show enough work.

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1. [15 points] Compute $\iint_R 2(x+1)y^2\,dA$, $R = [0,1] \times [0,3]$, by Riemann sum definition. You must use the Riemann sum definition to receive credit.

x: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

$x_i = 0 + i\Delta x = \frac{i}{n}$

y: $\Delta y = \frac{3-0}{m}$

$y_j = 0 + j\Delta y = 3\frac{j}{m}$

$f(x,y) = 2(x+1)y^2$

$f(x_i, y_j) = 2\left(\frac{i}{n}+1\right)\left(3\frac{j}{m}\right)^2$

$= 18\left(\frac{i}{n}+1\right)\left(\frac{j^2}{m^2}\right)$

$$\iint_R 2(x+1)y^2\,dA = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} 18\left(\frac{i}{n}+1\right)\left(\frac{j^2}{m^2}\right)\left(\frac{1}{n}\right)\left(\frac{3}{m}\right)$$

$$= 54 \lim_{m\to\infty}\left(\sum_{j=1}^{m}\frac{j^2}{m^3}\right)\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left[\frac{i}{n^2}+\frac{1}{n}\right]\right)$$

$$= 54\left(\lim_{m\to\infty}\frac{1}{m^3}\frac{m(2m+1)(m+1)}{6}\right)\left(\lim_{n\to\infty}\frac{1}{n^2}\frac{(n+1)n}{2}+\frac{n}{n}\right)$$

$$= 54\cdot\frac{1}{3}\cdot\left(\frac{1}{2}+1\right)$$

$$= 54\cdot\frac{1}{3}\cdot\frac{3}{2}$$

$$= \boxed{27}$$

2–4. [10 points each] Compute the double integrals. Show enough work.

2. $\displaystyle\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx$ $\displaystyle = \int_1^4 \left[x\ln y + \frac{y^2}{2x}\Big|_{y=1}^{2} \right] dx$

$$= \int_1^4 x\ln 2 + \frac{4}{2x} - \frac{1}{2x} \quad dx = \int_1^4 x\ln 2 + \frac{3}{2}\frac{1}{x}\,dx$$

$$= \frac{1}{2}\ln 2\cdot x^2 + \frac{3}{2}\ln x \Big|_{x=1}^{4}$$

$$= 8\ln 2 + \frac{3}{2}\ln 4 - \frac{1}{2}\ln 2$$

$$= 11\ln 2 - \frac{1}{2}\ln 2$$

$$= \boxed{\frac{21}{2}\ln 2}$$

3. $\displaystyle\iint_R \frac{xy^2}{x^2+1}\,dA,\quad R=[0,1]\times[-3,3]$

Separable!

$$\frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx \cdot \int_{-3}^{3} y^2\,dy = \left[\frac{1}{\cancel{2}}\ln(x^2+1)\Big|_0^1\right]\left[\frac{\cancel{2}y^3}{3}\Big|_0^3\right]$$

$$= (\ln 2 - \ln 1)(9) = \boxed{9\ln 2}$$

4. $\displaystyle\iint_R x\sin(x+y)\,dA,\quad R=[0,\pi/6]\times[0,\pi/3]$

<u>IbP</u>: $u = x \quad dv = \sin(x+y)\,dx$

$du = dx \quad v = -\cos(x+y)$

$$\rightarrow \int_0^{\pi/6} x\sin(x+y)\,dx = -x\cos(x+y)\Big|_0^{\pi/6} + \int_0^{\pi/6}\cos(x+y)\,dx = -x\cos(x+y) + \sin(x+y)\Big|_{0=x}^{\pi/6}$$

$$= \frac{-\pi}{6}\cos(\pi/6+y) + \sin(\pi/6+y) - \left[0 + \sin y\right]$$

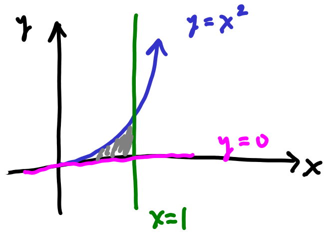
and $\displaystyle\iint_R x\sin(x+y)\,dA = \int_0^{\pi/3}\left[\frac{-\pi}{6}\cos(\pi/6+y) + \sin(\pi/6+y) - \sin y\right]dy = -\frac{\pi}{6}\sin(\pi/6+y) - \cos(\pi/6+y) + \cos y\Big|_0^{\pi/3}$

$$= \frac{-\pi}{6}\underbrace{\sin(\pi/2)}_{1} - \underbrace{\cos(\pi/2)}_{0} + \underbrace{\cos(\pi/3)}_{1/2} + \frac{\pi}{6}\underbrace{\sin(\pi/6)}_{1/2} + \underbrace{\cos(\pi/6)}_{\sqrt{3}/2} - \underbrace{\cos(0)}_{1} = \boxed{\frac{\sqrt{3}-1}{2} - \frac{\pi}{12}}$$

5. [15 points] Evaluate the double integral

$$\iint_D x\cos y\, dA,$$

where D is bounded by $y=0$, $y=x^2$, and $x=1$.



Type I: $y: 0 \to x^2$
$x: 0 \to 1$

$$\iint_D x\cos y\, dA = \int_0^1\int_0^{x^2} x\cos y\, dy\, dx$$

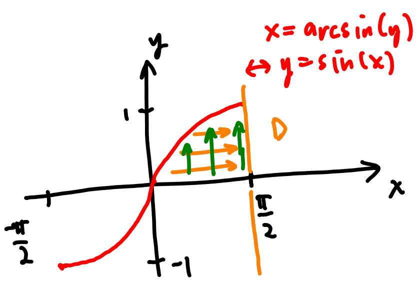
$$= \int_0^1 x\sin y \Big|_0^{x^2} dx$$

$$= \frac{1}{2}\int_0^1 2x\sin(x^2)\, dx$$

$$= \frac{1}{2}\left(-\cos x^2\Big|_0^1\right) = \frac{1}{2}\left(-\cos(1)+1\right)$$

$$= \boxed{\frac{1}{2}\left(1-\cos(1)\right)}$$

6. [15 points] Evaluate the integral by (carefully!) reversing the order of integration.



$$\int_0^1\int_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\; dx\, dy$$

Current: type II: $\begin{cases} x: \arcsin y \to \pi/2 \\ y: 0 \to 1 \end{cases}$

Change: type I: $\begin{cases} y: 0 \to \sin x \\ x: 0 \to \pi/2 \end{cases}$

$$\int_0^1\int_{\arcsin y}^{0} \cos x\sqrt{1+\cos^2 x}\, dx\, dy = \int_0^{\pi/2}\int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\, dy\, dx$$

$$= -\frac{1}{2}\int_0^{\pi/2} -2\sin x\cos x\sqrt{1+\cos^2 x}\, dx$$

$u = 1+\cos^2 x$ $\quad u(\pi/2) = 1$

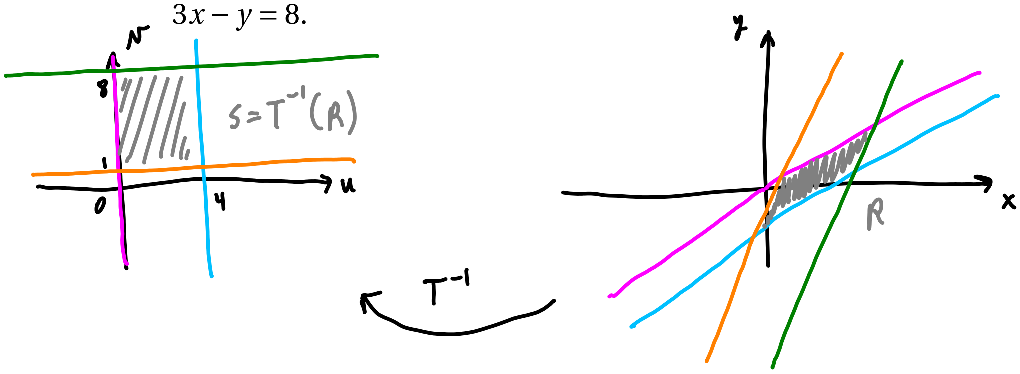
$du = -2\cos x\sin x$ $\quad u(0) = 2$

$$= \frac{1}{2}\int_1^2 \sqrt{u}\, du = \frac{1}{3}u^{3/2}\Big|_1^2 = \frac{\sqrt{8}}{3} - \frac{1}{3} = \boxed{\frac{\sqrt{8}-1}{3}}$$

7. [25 points] Evaluate the integral by making an appropriate change of variables.

$$\iint_R \frac{x-2y}{3x-y}\,dA,$$

where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$.



$x-2y=0 \to y=\frac{1}{2}x$

$x-2y=4 \to y=\frac{1}{2}x-2$

$3x-y=1 \to y=3x-1$

$3x-y=8 \to y=3x-8$

$T^{-1}: \begin{cases} u = x-2y \\ v = 3x-y \end{cases}$ $\quad u: 0 \to 4$, $\quad v: 1 - 8$

T : solve for (x,y). $\quad \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} \to A^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$, so $\boxed{\begin{cases} x = \frac{1}{5}(-u+2v) \\ y = \frac{1}{5}(-3u+v) \end{cases}}$

Jacobian: $\quad J(u,v) = \det\begin{pmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{pmatrix} = -\frac{1}{5^2} + \frac{6}{5^2} = \frac{5}{5^2} = \frac{1}{5}$

The integral becomes:

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_1^8 \int_0^4 \frac{u}{v}\cdot\frac{1}{5}\,du\,dv = \frac{1}{5}\cdot\int_1^8 \frac{1}{v}\,dv \cdot \int_0^4 u\,du = \frac{1}{5}\cdot \ln 8 \cdot \frac{4^2}{2} = \boxed{\frac{8}{5}\ln 8}$$ or

$$= \boxed{\frac{24}{5}\ln 2}$$

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