Name: Key

M344: Calculus III (Su.19)

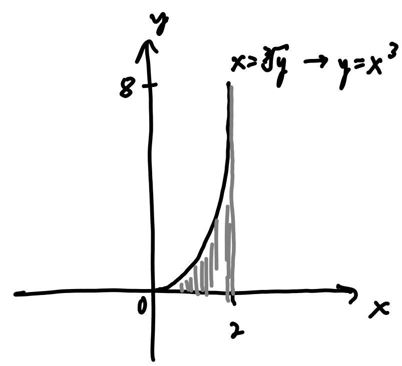
Good Problems 5

Selections from Chapter 15

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use your own hand-written notes, but you may not use any electronic devices.

1. [25 points] Compute the double integral by reversing the order of integration.

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$$



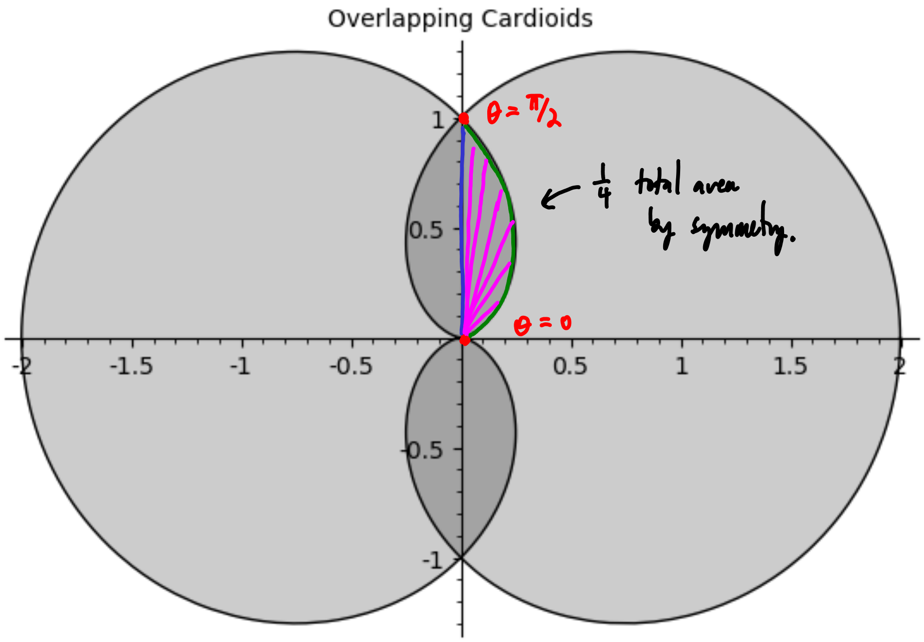
$$= \int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx$$

$$= \frac{1}{4}\int_0^2 4x^3 e^{x^4}\, dx$$

$$= \frac{1}{4} e^u \Big|_0^{16} = \boxed{\frac{1}{4}\left(e^{16}-1\right)}$$

2. [25 points] Use a double integral and symmetry to find the area of the region bounded between the cardiods

$$\begin{cases} r(\theta) = 1 + \cos\theta, & \text{and} \\ r(\theta) = 1 - \cos\theta. \end{cases}$$



$$A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} 1 \, dA = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r \, dr \, d\theta$$

$$= 4\int_0^{\pi/2} \frac{1}{2}r^2\Big|_0^{1-\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta$$

$$= 2\int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} 1 \, d\theta - \int_0^{\pi/2} \cos\theta \, d\theta + \int_0^{\pi/2} 1 + \cos 2\theta \, d\theta$$

$$= \pi - (1-0) + \pi/2 = \frac{3\pi}{2} - 1$$

$$= \boxed{\frac{3\pi - 2}{2}}$$

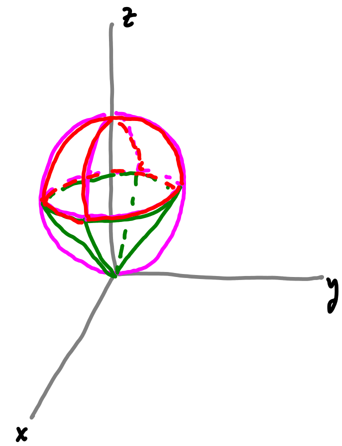
3. [30 points] A solid lies above the cone

$$z = \sqrt{x^2 + y^2}$$

and below the sphere

$$x^2 + y^2 + z^2 = z.$$

Use spherical coordinates to compute the volume of this solid.



Sphere: $\rho^2 = \rho \cos\varphi \rightarrow \rho = \cos\varphi$

$$\begin{cases} \rho : 0 \rightarrow \cos\varphi \\ \varphi : 0 \rightarrow \pi/4 \\ \theta : 0 \rightarrow 2\pi \end{cases}$$

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \int_0^{\pi/4} \frac{1}{3} \cdot \cos^3\varphi \sin\varphi \, d\varphi$$

$u = \cos\varphi$ $\quad u(\pi/4) = \sqrt{2}/2$

$du = -\sin\varphi \, d\varphi$ $\quad u(0) = 1$

$$= \frac{2\pi}{3} \int_{\frac{\sqrt{2}}{2}}^{1} u^3 \, du = \frac{\pi}{6}\left(1 - 1/4\right) = \pi/6 \cdot 3/4 = \boxed{\frac{\pi}{8}}$$

4. [30 points] Use a change-of-coordinates transformation to compute the integral

$$\iint_R \sin\left(9x^2 + 4y^2\right)\, dA$$

where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$.

$$9x^2 + 4y^2 = (3x)^2 + (2y)^2 = u^2 + v^2 = 1.$$

$$T^{-1}: \begin{cases} u = 3x \\ v = 2y \end{cases} \rightarrow T: \begin{cases} x = \tfrac{1}{3}u \\ y = \tfrac{1}{2}v \end{cases}$$

<u>Jacobian</u>: $J(u,v) = \begin{vmatrix} 1/3 & 0 \\ 0 & 1/2 \end{vmatrix} = \frac{1}{6}$

$$\iint_R \sin(9x^2 + 4y^2)\, dA = \frac{1}{6} \iint_S \sin(u^2 + v^2)\, dA$$

$$= \frac{1}{6} \int_0^{\pi/2} \int_0^1 \sin(r)\, r\, dr\, d\theta$$

$$= \frac{\pi}{12} \int_0^1 r \sin(r)\, dr$$

IbP:

u	dv
$+\ r$	$\sin(r)$
$-\ 1$	$-\cos(r)$
$+\ 0$	$-\sin(r)$

$$= \frac{\pi}{12}\left[-r\cos(r) + \sin(r)\right]\Big|_0^1$$

$$= \frac{\pi}{12}\left[-\cos(1) + \sin(1) - \sin 0\right]$$

$$= \boxed{\frac{\pi\,(\sin(1) - \cos(1) - 1)}{12}}$$

scratch page