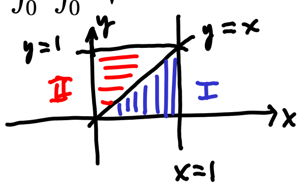
Name: Key
M344: Calculus III (Su.19)
Extra Credit: Integration
Due by: Friday, 12 July 2019 at 8:40 am

Instructions. Complete all problems on this paper, showing enough work. You may use any resources that you'd like to complete this assignment, but be sure to write your solutions in your own words.

This assignment is optional. If submitted, it will be counted as extra credit toward your final grade.

1–4. True/False [1 point each] Write a **T** on the line if the statement is always true, and **F** otherwise. If you determine that the statement is false, you must give justification in the space provided to receive credit.

F **1.** $\overbrace{\int_0^1 \int_0^x \sqrt{x+y^2}\,dy\,dx}^{\mathrm{II}} = \overbrace{\int_0^1 \int_0^y \sqrt{x+y^2}\,dx\,dy}^{\mathrm{I}}$



The regions do not coincide.

T **2.** If f is continuous on $[0,1]$, then $\int_0^1 \int_0^1 f(x)f(y)\,dy\,dx = \left[\int_0^1 f(x)\,dx\right]^2$.

This is separable: $\int_0^1 \int_0^1 f(x)f(y)\,dy\,dx = \int_0^1 f(x)dx \cdot \int_0^1 f(y)dy = \left[\int_0^1 f(x)dx\right]^2$

Put $y=x$
$dy=dx$

T **3.** $\overbrace{\int_1^4 \int_0^1 (x^2+\sqrt{y})\sin(x^2y^2)\,dx\,dy}^{\mathrm{I}} \le 9$

$\sin(x^2y^2) \le 1$, so $I \le \int_1^4 \int_0^1 x^2+\sqrt{y}\,dx\,dy$

$x^2 \le 1$ on $[0,1]$, so $I \le \int_1^4 1+\sqrt{y}\,dy \cdot (1-0) = \int_1^4 1+\sqrt{y}\,dy$

$1+\sqrt{y} \le 1+2=3$ on $[1,4]$, so $I \le 3\cdot(4-1) = 3\cdot 3 = 9$. ∎

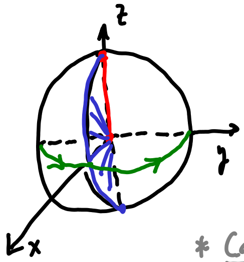
F **4.** The integral $\int_0^{2\pi} \int_0^2 \int_r^2 dz\,dr\,d\theta$ represents the volume enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z=2$.

The region is correct, but dV must be "$r\,dz\,dr\,d\theta$".

5. [2 points] Compute the integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy.$

Smells like spherical.

"Front" half of a sphere w $\rho \le 2$.
$(x \ge 0.)$



$\rho: 0 \to 2$
$\varphi: 0 \to \pi$
$\theta: 0 \to \frac{\pi}{2}$
$\frac{3\pi}{2} \to 2\pi$

* Caution: Changing this could change the Jacobian! — Which is fine, but you'd have to compute it.

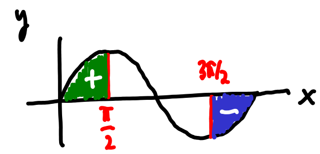
$y^2\sqrt{x^2+y^2+z^2} = \rho^3 \sin\varphi \sin\theta$

$dz\,dx\,dy = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

so the integrand is:

$$\int_0^{\pi/2}\int_0^{\pi}\int_0^{2} \rho^5 \sin^2\varphi \sin\theta\, d\rho\, d\varphi\, d\theta + \int_{3\pi/2}^{2\pi}\int_0^{\pi}\int_0^{2} \rho^5 \sin^2\varphi \sin\theta\, d\rho\, d\varphi\, d\theta$$

$$= \left(\int_0^{\pi/2} \sin\theta\, d\theta + \int_{3\pi/2}^{2\pi} \sin\theta\, d\theta\right)\cdot \int_0^{\pi} \sin^2\varphi\, d\varphi \cdot \int_0^{2} \rho^5\, d\rho$$



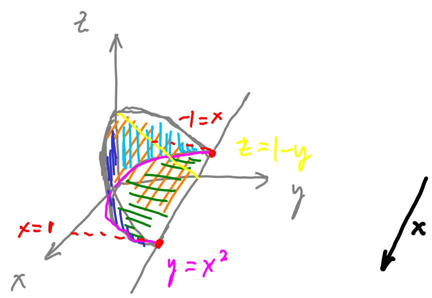
$$= 0 \cdot \int_0^{\pi} \sin^2\varphi\, d\varphi \cdot \int_0^{2} \rho^5\, d\rho = \boxed{0}.$$

6. [2 points] Rewrite the integral $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$ as an iterated integral in the order $dx\,dy\,dz$.

Current:

$z: 0 \to 1-y$
$y: x^2 \to 1$
$x: -1 \to 1$

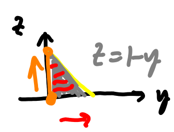
Sketch:



New order:

$x: -\sqrt{y} \to \sqrt{y}$
$y: 0 \to 1-z$
$z: 0 \to 1$

→

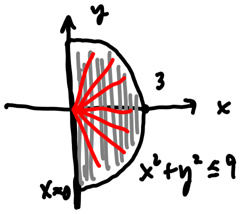


New integral:

$$\boxed{\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\, dx\, dy\, dz}$$

7. [2 points] Evaluate the integral $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\ dy\,dx$.

Polar!



$r: 0 \to 3$

$\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$

$x^3 = r^3\cos^3\theta$

$xy^2 = r^3\cos\theta\sin^2\theta$

The integral becomes:

$$\int_{-\pi/2}^{\pi/2}\int_0^3 \left(r^3\cos\theta\cos^2\theta + r^3\cos\theta\sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta \cdot \int_0^3 r^4\,dr$$

$$= 2\int_0^{\pi/2}\cos\theta\,d\theta \cdot \tfrac{1}{4}r^4\Big|_0^3$$

$$= \left(2\sin\tfrac{\pi}{2} - 2\sin 0\right)\cdot\frac{81}{4} = \boxed{\frac{81}{2}}$$

8. [2 points] Compute the Jacobian of the spherical coordinate transformation. Show enough work.

$$\left.\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned}\right\}$$

Expand

$$J = \det\begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \end{pmatrix} \begin{matrix} + \\ - \\ + \end{matrix}$$

$$= \cos\varphi\left(\rho^2\cos\varphi\sin\varphi\cos^2\theta + \rho^2\cos\varphi\sin\varphi\sin^2\theta\right)$$

$$+ \rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right)$$

$$+ 0$$

$$= \rho^2\sin\varphi\cos^2\varphi\,\big(\overbrace{\cancel{\cos^2\theta+\sin^2\theta}}^{1}\big) + \rho^2\sin\varphi\sin^2\varphi\,\big(\overbrace{\cancel{\cos^2\theta+\sin^2\theta}}^{1}\big)$$

$$= \rho^2\sin\varphi\,\big(\overbrace{\cancel{\cos^2\varphi+\sin^2\varphi}}^{1}\big)$$

$$= \boxed{\rho^2\sin\varphi} = J(\rho,\varphi,\theta)$$

9. [2 points] Read about centroids and centers of mass in section 15.4 of the book.

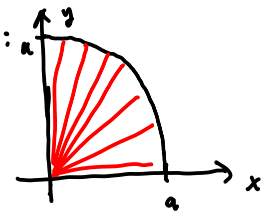
A lamina occupies the part of the disk $x^2 + y^2 \leq a^2$, $a > 0$, that lies in the first quadrant. Find

1. the centroid of the lamina; and
2. the center of mass of the lamina if the desity function is $\rho(x, y) = xy^2$.

Formulas: $m = \iint_D \rho(x,y)\,dA$, $\quad M_y = \iint_D x\rho(x,y)\,dA$, $\quad M_x = \iint_D y\rho(x,y)\,dA$

$$\bar{x} = \frac{M_x}{m} \text{ and } \bar{y} = \frac{M_y}{m}.$$

Region:



$r: 0 \to a$

$\theta: 0 \to \pi/2$

$dA = r\,dr\,d\theta$

1. $\rho = 1$.

$$m = \int_0^{\pi/2}\int_0^a r\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{2}a^2\,d\theta = \frac{a^2\pi}{4}$$

$$M_x = \int_0^{\pi/2}\int_0^a r^2 \sin\theta\,dr\,d\theta = \left(-\cos\theta\Big|_0^{\pi/2}\right)\left(\frac{1}{3}r^3\Big|_0^a\right) = \frac{1}{3}a^3$$

$$M_y = \int_0^{\pi/2}\int_0^a r^2 \cos\theta\,dr\,d\theta = \left(\sin\theta\Big|_0^{\pi/2}\right)\left(\frac{1}{3}r^3\Big|_0^a\right) = \frac{1}{3}a^3$$

So the centroid is $\left(\frac{Mx}{m}, \frac{My}{m}\right) = \left(\frac{a^3}{3}\cdot\frac{4}{a^2\pi}, \frac{a^3}{3}\cdot\frac{4}{a^2\pi}\right) = \boxed{\left(\frac{4}{3}\frac{a}{\pi}, \frac{4}{3}\frac{a}{\pi}\right) = (\bar{x}, \bar{y})}$

2. $\rho = xy^2$

$$m = \int_0^{\pi/2}\int_0^a r^3\cos\theta\sin^2\theta \cdot r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\sin^2\theta\,d\theta \cdot \int_0^a r^4\,dr = \frac{1}{3}\cdot\frac{1}{5}a^5 = \frac{1}{15}a^5$$

$$M_x = \int_0^{\pi/2}\int_0^a r^4\cos\theta\sin^3\theta\, r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\sin^3\theta\,d\theta \cdot \int_0^a r^5\,dr = \frac{1}{4}\cdot\frac{1}{6}a^6 = \frac{1}{24}a^6$$

$$M_y = \int_0^{\pi/2}\int_0^a r^4\cos^2\theta\sin^2\theta\, r\,dr\,d\theta = \int_0^{\pi/2}(\cos\theta\sin\theta)^2\,d\theta \cdot \int_0^a r^5\,dr = \int_0^{\pi/2}\left(\frac{1}{2}\sin(2\theta)\right)^2 d\theta \cdot \frac{1}{6}a^6$$

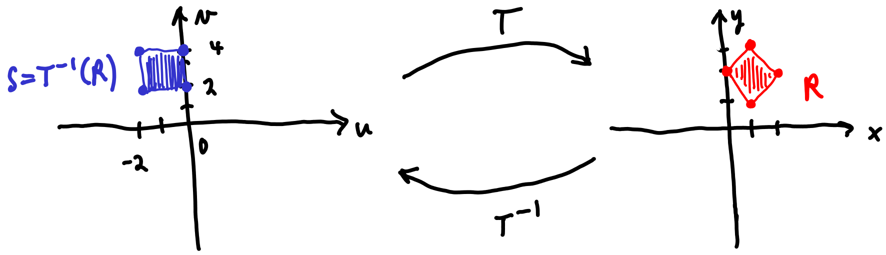
$$= \frac{a^6}{48}\int_0^{\pi/2} 1 - \cos(4\theta)\,d\theta = \frac{\pi a^6}{96}$$

So the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{a^6}{24}\cdot\frac{15}{a^5}, \frac{\pi a^6}{96}\cdot\frac{15}{a^5}\right) = \boxed{\left(\frac{5a}{8}, \frac{5\pi a}{32}\right)}$

10. [3 points] Use a change-of-coordinates transformation to compute the integral

$$\iint_R \frac{x-y}{x+y}\,dA,$$

where R is the square with vertices $(0,2)$, $(1,1)$, $(2,2)$, and $(1,3)$.



$T^{-1}: \begin{cases} u = x-y \\ v = x+y \end{cases} : \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, so $T: \begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = -\frac{1}{2}u + \frac{1}{2}v \end{cases}$

$J(u,v) = \det(A^{-1}) = \frac{1}{4}(1+1) = \frac{2}{4} = \frac{1}{2}.$

So, $\iint_R \frac{x-y}{x+y}\,dA = \frac{1}{2}\int_2^4 \int_{-2}^0 \frac{u}{v}\,du\,dv = \frac{1}{2}\int_2^4 \frac{1}{v}\,dv \cdot \int_{-2}^0 u\,du = \frac{1}{2}\left(\ln(4) - \ln(2)\right)\cdot\left(\frac{0^2}{2} - \frac{(-2)^2}{2}\right)$

$= \frac{1}{2}\left(2\ln 2 - \ln 2\right)\cdot(-2)$

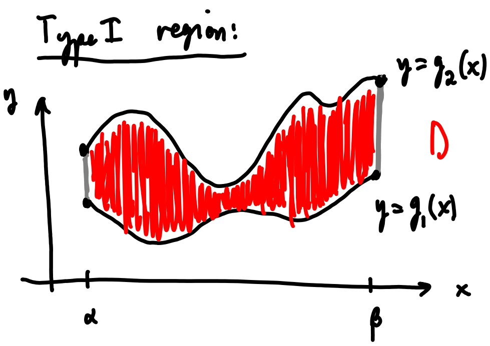
$= \boxed{-\ln 2}$

or $= \boxed{\ln(1/2)}$

11. [3 points] Let D be a type I region in the xy-plane, and suppose f is a continuous function on D. Prove that there exists a point (x_0, y_0) in D satisfying

$$\iint_D f(x,y)\,dA = f(x_0,y_0)\cdot \text{Area}(D).$$

This is the **Mean Value Theorem** for double integrals.



$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$= \int_\alpha^\beta \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$