Computational Geometry Workshop

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Vector Fields and 1-Forms

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This notebook contains sample code related to the lecture on Differential Forms. Terse lecture notes may be found at

http://geometerjustin.com/teaching/cgw/notes/notes_01.html (http://geometerjustin.com/teaching/cgw/notes/notes_01.html)

So future readers know what version we're using:

In [1]:
```
version()
```

Out[1]: 'SageMath version 7.5.1, Release Date: 2017-01-15'

Get outputs in LaTeX form

In [2]:
```
%display latex
```

We begin by defining the exterior algebra of forms over a coordinate chart U in R^n. The best way to do this is to define U to be a 3-dimensional real manifold with a single chart. (We will define manifolds in full rigor later in the semester.)

In [3]:
```
U = Manifold(3,'U',latex_name=r'\mathbb{U}',start_index=1)
```

In [4]:
```
print(U)
U
```

```
3-dimensional differentiable manifold U
```

Out[4]: $\mathbb{U}$

Next we define the coordinates.

In [5]:
```
coord.<x,y,z>=U.chart()
```

In [6]:
```
print(coord)
coord
```

```
Chart (U, (x, y, z))
```

Out[6]: $(\mathbb{U}, (x, y, z))$

In [7]:
```
p = U.point((1,-1,3),chart=coord);
print(p)
```

```
Point on the 3-dimensional differentiable manifold U
```

Defining coordinates automatically defines a basis for the tangent bundle to U. This is called a *frame field* on U.

In [8]:
```
coord.frame()
```

Out[8]: $\left(\mathbb{U}, \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right)$

A vector field is an \F-linear combination of these basis vectors.

In [9]:
```
X1 = U.vector_field(name='X1',latex_name=r'X_1')
```

In [10]:
```
X1[:]=[x^2,-y,2*x*y-z]
```

In [11]:
```
print(X1)
X1
```

```
Vector field X1 on the 3-dimensional differentiable manifold U
```

Out[11]: X_1

In [12]:
```
X1.display()
```

Out[12]:

$$X_1 = x^2 \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + (2\,xy - z) \frac{\partial}{\partial z}$$

Vector fields act on smooth functions. We must regard a function on U as a scalar field on U.

In [13]:
```
f = U.scalar_field({coord:x+2*y-z})
f.display()
```

Out[13]:

$$\begin{array}{llcl} & \mathbb{U} & \longrightarrow & \mathbb{R} \\ & (x, y, z) & \longmapsto & x + 2\,y - z \end{array}$$

In [14]:
```
f(p)
```

Out[14]: -4

We can now apply the vector field to the function

In [15]:
```
X1(f).display()
```

Out[15]:

$$\begin{array}{llcl} & \mathbb{U} & \longrightarrow & \mathbb{R} \\ & (x, y, z) & \longmapsto & x^2 - 2\,(x + 1)y + z \end{array}$$

This should be the same answer that we get by applying the differential of f to X1, as in Calculus III

In [16]:
```
f.differential()(X1).display()
```

Out[16]:

$$\begin{array}{llcl} & \mathbb{U} & \longrightarrow & \mathbb{R} \\ & (x, y, z) & \longmapsto & x^2 - 2\,(x + 1)y + z \end{array}$$

In [17]:
```
X1(f) == f.differential()(X1)
```

Out[17]: True

In [18]:
```
X1(f)(p)
```

Out[18]: 8

We can also easily compute the Lie bracket of two vector fields

In [19]:
```
X2 = U.vector_field(name='X2',latex_name=r'X_2');
X2[:] = [0,3*x*y,tan(x*z)];
X1X2 = X1.bracket(X2);
print(X1X2)
X1X2.display()
```

Vector field on the 3-dimensional differentiable manifold U

Out[19]:
$$3\,x^2 y \frac{\partial}{\partial y} + \left(-\frac{6\,x^2 y \cos{(xz)}^2 - 2\,x^2 y - \left(x^2 - x\right) z - \cos(xz)\sin(xz)}{\cos{(xz)}^2} \right) \frac{\partial}{\partial z}$$

Choosing coordinates on U also automatically defines a frame field for the cotangent bundle. This is called a *coframe* on U.

In [20]:
```
coord.coframe()
```

Out[20]: $(\mathbb{U}, (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z))$

A 1-form is an \F-linear combination of these basis covectors.

In [21]:
```
t1 = U.diff_form(1,name='t1',latex_name=r'\theta_1')
```

In [22]:
```
t1[:] = [x,1/y,sin(z)]
```

In [23]:
```
print(t1)
t1
```

1-form t1 on the 3-dimensional differentiable manifold U

Out[23]: θ_1

In [24]:
```
t1.display()
```

Out[24]:
$$\theta_1 = x\mathrm{d}x + \frac{1}{y}\mathrm{d}y + \sin(z)\mathrm{d}z$$

1-forms act on vector fields.

In [25]:
```
t1(X1)
```

Out[25]: $\theta_1\left(X_1\right)$

In [26]:
```
t1(X1).display()
```
Out[26]:

$$\theta_1\left(X_1\right):\ \begin{array}{llcl} & \mathbb{U} & \longrightarrow & \mathbb{R} \\ & (x, y, z) & \longmapsto & x^3 + (2\,xy - z)\sin(z) - 1 \end{array}$$

In [27]:
```
t1(X1)(p)
```
Out[27]:

$$-5\ \sin(3)$$

In [ ]:
```
```