**Name:** Key

**M555: Differential Equations I** (Spring 2018)
Instructor: Justin Ryan
Good Problems 2 – Sections 2.1, 2.2, 2.4

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**Instructions** *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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**1.** Find the particular solution of the initial value problem.

$$\begin{cases} ty' + (t+1)y = t, \\ y(\ln 2) = 1, \quad t > 0 \end{cases}$$

$y' + \frac{t+1}{t} y = 1$ Linear FODE!

$p(t) = \frac{t+1}{t} = 1 + \frac{1}{t}$

$\int p(t)\,dt = t + \ln t$

$\mu(t) = e^{t} e^{\ln t} = t e^{t}$

Thus, the solution is

$$y = \frac{1}{t}e^{-t}\left(\int_{\ln 2}^{t} u e^{u}\,du + 1\right) = \frac{1}{t}e^{-t}\left(ue^{u} - e^{u}\Big|_{\ln 2}^{t} + 1\right)$$

$$= \frac{1}{t}e^{-t}\left(te^{t} - e^{t} - 2\ln 2 + 2 + 1\right)$$

$$\boxed{y = 1 - \frac{1}{t} + (3 - 2\ln 2)\frac{1}{t}e^{-t}}$$

**2.** Use the method of variation of parameters (see exercise 2.1.38) to find the general solution to the differential equation. You <u>must</u> use variation of parameters to receive credit.

$$y' - 2y = t^2 e^{2t}$$

To use variation of parameters, first compute $e^{-\int p\,dt} = e^{-\int -2\,dt} = e^{2t} = \frac{1}{\mu(t)}$

Then assume the solution of the DE is of the form

$$y = A(t)\, e^{2t}$$

where $A$ must satisfy the DE,

$$A' = \overbrace{t^2 e^{2t}}^{g(t)} \cdot \overset{\mu(t)}{e^{-2t}} = t^2$$

Therefore, $A = \frac{1}{3}t^3 + C$ and the solution is

$$\boxed{y = \frac{1}{3}t^3 e^{2t} + Ce^{2t}}$$

**3.** Solve the initial value problem ~~and determine the interval on which the solution is valid.~~

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{1+3x^2}{3y^2-6y}, \\ y(0) = 1 \end{cases}$$

Don't worry about this part of the question for this DE.

This equation is separable!

$$(3y^2-6y)\,dy = (1+3x^2)\,dx$$

To solve, integrate

$$\int_1^y 3u^2 - 6u\,du = \int_0^x 1+3v^2\,dv$$

$$u^3 - 3u^2 \Big|_1^y = v + v^3 \Big|_0^x$$

$$y^3 - 3y^2 - (1-3) = x + x^3$$

$$\Rightarrow \boxed{y^3 - 3y^2 - x^3 - x = -2}$$

This "can" be solved for y, but it's a mess, so leave it like this.

**4.** Verify that $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are both solutions of the initial value problem

$$\begin{cases} y' = \dfrac{-t + \sqrt{t^2 + 4y}}{2}, \\ y(2) = -1. \end{cases}$$

Explain why the existence of two different solutions does not contradict the Fundamental Existence and Uniqueness Theorem.

a.) $y_1' = -1$ : $\dfrac{-t + \sqrt{t^2 + 4(1-t)}}{2} = \dfrac{-t + \sqrt{t^2 - 4t + 4}}{2} = \dfrac{-t + \sqrt{(t-2)^2}}{2} = \dfrac{-t + t - 2}{2} = \dfrac{-2}{2} = -1.$ ✓

$y_2' = -\frac{t}{2}$ : $\dfrac{-t + \sqrt{t^2 + 4(-t^2/4)}}{2} = \dfrac{-t + \sqrt{t^2 - t^2}}{2} = -\dfrac{t + 0}{2} = -\dfrac{t}{2}$ ✓

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$f(t,y) = \dfrac{-t + \sqrt{t^2 + 4y}}{2}$ is continuous for $t^2 \geq -4y$

$\dfrac{\partial f}{\partial y} = -\dfrac{t}{2} + \dfrac{4}{4\sqrt{t^2 + 4y}}$ is continuous for $t^2 > -4y$

The given initial condition satisfies $t^2 = -4y$. Therefore the FEUT does not apply. Hence the lack of uniqueness of the solution.