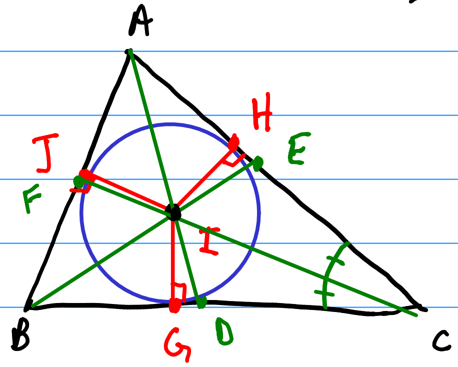
M621 9/12/18

§1.8 – The Incircle and Excircles



AD, BE, CF are angle bisectors

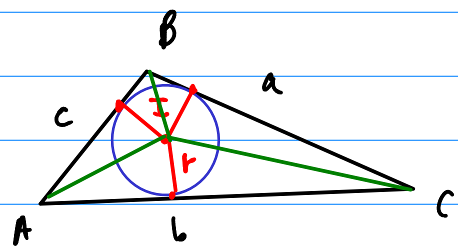
Thm. 1.8.1 The angle bisectors of a triangle $\triangle ABC$ are concurrent and they intersect at the center of the incircle.

"Proof". By similar triangle arguments. $\triangle JAI \sim \triangle HAI$, etc.

Thm 1.8.2. Let r be the radius of the inscribed circle of a triangle $\triangle ABC$ and let $s = \frac{1}{2}(a+b+c)$ be the semiperimeter. Then

$$\text{Area}(\triangle ABC) = |\triangle ABC| = rs$$

Sketch.



Proof. $|\triangle ABC| = |\triangle AIC| + |\triangle AIB| + |\triangle BIC|$

$$= \tfrac{1}{2}br + \tfrac{1}{2}cr + \tfrac{1}{2}ar$$

$$= r \cdot \tfrac{1}{2}(a+b+c)$$

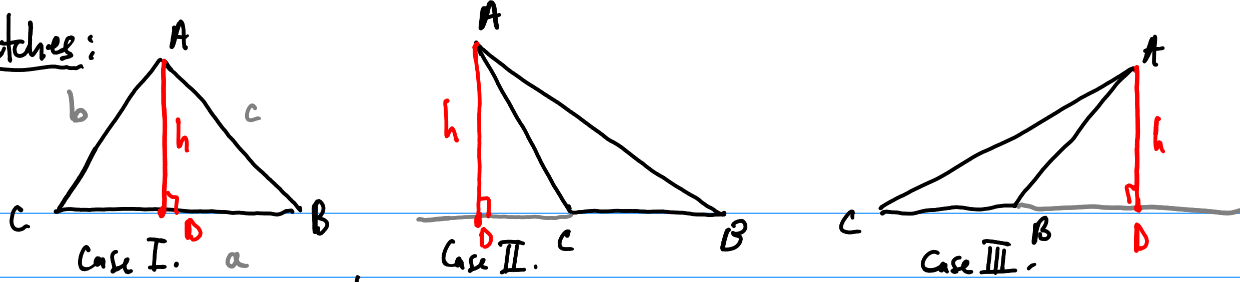
$$= rs \quad \blacksquare$$

Thm. 1.8.3 (Law of Cosines)

Given $\triangle ABC$ w/ (opposite) side lengths a, b, c, then

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Sketches:



Focus on case I

Proof. By Pythagorean Thm, $c^2 = (AD)^2 + (DB)^2$

$\sin(C) = \frac{AD}{b} \Rightarrow AD = b \sin C$

$DB = a - DC$

$\cos(C) = \frac{DC}{b} \Rightarrow DC = b \cos C$

$DB = a - b\cos C$

Now, $c^2 = (b \sin C)^2 + (a - b\cos C)^2$

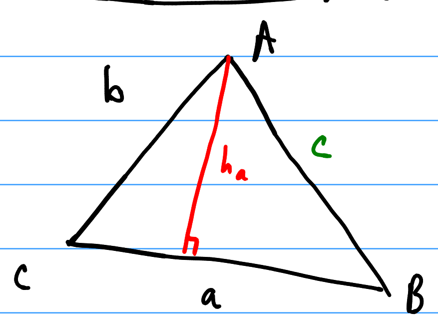
$c^2 = b^2 \sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C$

$c^2 = a^2 + b^2(\sin^2 C + \cos^2 C) - 2ab\cos C$

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

∎

Formula for area of triangle: ΔABC, $|\Delta ABC| = \frac{1}{2} bh$.



$|\Delta ABC| = \frac{1}{2} a h_a$ but $h_a = b \sin C$ by our previous work.

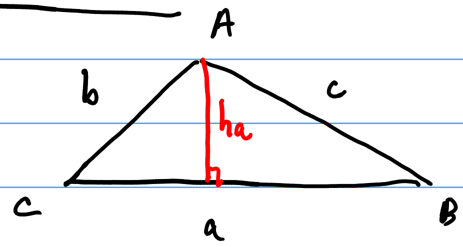
$$\boxed{|\Delta ABC| = \frac{1}{2} ab \sin C}$$

or $h_a = c \sin B$

$$\boxed{|\Delta ABC| = \frac{1}{2} ac \sin B}$$

$$\boxed{|\Delta ABC| = \frac{1}{2} bc \sin A}$$

Thm (Law of Sines)



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\sin C = \frac{h_a}{b} \Rightarrow h_a = b \sin C$$

$$\sin B = \frac{h_a}{c} \rightarrow h_a = c \sin B$$

$$h_a = \underbrace{b \sin C = c \sin B}$$

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

Similarly, we get $\frac{\sin A}{a} = \frac{\sin B}{b} \ldots$ etc.

Thm 1.8.4 (Heron's Formula for Area).

ΔABC, $s = \frac{1}{2}(a+b+c)$.

$$|\Delta ABC| = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin^2 C + \cos^2 C = 1$$
$$\sin^2 C = 1 - \cos^2 C$$
$$\sin C = \sqrt{1-\cos^2 C}$$

Idea of Proof: $|\Delta ABC| = \frac{1}{2} ab \sin C$
$$= \frac{1}{2} ab \sqrt{1-(\cos C)^2}$$

$$|\Delta ABC|^2 = \frac{1}{4} a^2 b^2 (1 - \cos^2 C)$$

Recall: $c^2 = a^2 + b^2 - 2ab \cos C$ $\leftarrow$ solve for $\cos C$

$$\cos C = \frac{c^2 - a^2 - b^2}{-2ab} = \frac{a^2 + b^2 - c^2}{2ab}$$

Plugging in,

$$|\Delta ABC|^2 = \frac{1}{4}a^2b^2\left(1 - \frac{(a^2+b^2-c^2)^2}{4a^2b^2}\right)$$

$$= \frac{1}{16}\left(4a^2b^2 - (a^2+b^2-c^2)^2\right)$$

Multiply out
combine like terms
Factor carefully looking for
terms w/ $2s = a+b+c$

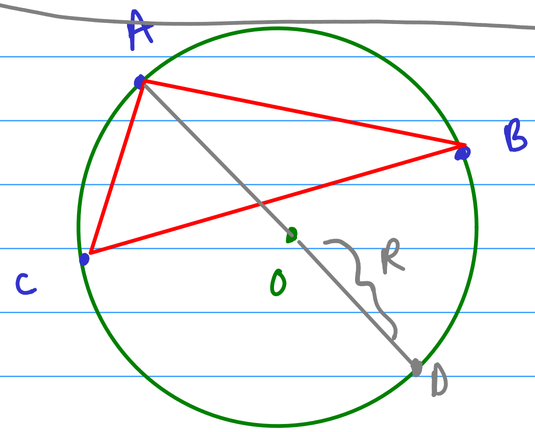
$$|\Delta ABC|^2 = s(s-a)(s-b)(s-c)$$

$$\boxed{|\Delta ABC| = \sqrt{s(s-a)(s-b)(s-c)}}$$

Thm. 1.8.5. The Extended Law of Sines

ΔABC w/ circumradius R,

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$$



$2R = AD$

Proof: RE.