Name:________________________________

M511: Linear Algebra (Spring 2018)

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Good Problems 1 – Section 3.1

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Let V be a vector space. Show that the zero element $\mathbf{0} \in V$ is unique.

2. Let V be a vector space and let $\mathbf{x} \in V$. Show that

1. $\beta\mathbf{0} = \mathbf{0}$ for all $\beta \in \mathbb{R}$, and
2. if $\alpha\mathbf{x} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{x} = \mathbf{0}$.

3. Let $\mathbb{R}^+$ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by

$$\alpha \circ x = x^\alpha$$

for each $x \in \mathbb{R}^+$ and each $\alpha \in \mathbb{R}$. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y$$

for all $x, y \in \mathbb{R}^+$. Is $(\mathbb{R}^+, \oplus, \circ)$ a vector space? Prove or disprove your answer.

4. Let $\mathbb{Z}$ denote the set of all integers with addition defined in the usual way, and define scalar multiplication, denoted $\circ$, by

$$\alpha \circ k = [\![\alpha]\!] \cdot k$$

for all $\alpha \in \mathbb{R}$ and all $k \in \mathbb{Z}$, where $[\![\ ]\!]$ is the greatest integer function. The space $(\mathbb{Z}, +, \circ)$ is not a vector space. Which axioms fail to hold, and why?

5. Consider the vector spaces $\mathbb{P}_n$ and $\mathbb{R}^n$ with their usual additions and scalar multiplications, and consider the correspondence

$$p(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1} \leftrightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \mathbf{a}.$$

Show that

1. $\alpha p \leftrightarrow \alpha \mathbf{a}$ for every $\alpha \in \mathbb{R}$, and
2. $p + q \leftrightarrow \mathbf{a} + \mathbf{b}$.