Name: Key

M511: Linear Algebra (Spring 2018)
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Good Problems 1 – Section 3.1

WICHITA STATE UNIVERSITY

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Let V be a vector space. Show that the zero element $\mathbf{0} \in V$ is unique.

Suppose $\bar{y} \in V$ satisfies $\bar{x} + \bar{y} = \bar{x}$ for all $\bar{x} \in V$. (*)

Therefore,

$$\begin{aligned} \bar{0} &= \bar{0} + \bar{y} && \text{by (*)} \\ &= \bar{y} + \bar{0} && \text{by A1.} \\ &= \bar{y} && \text{by A3.} \end{aligned}$$

and $\bar{0} \in V$ is unique. ◻

2. Let V be a vector space and let $\mathbf{x} \in V$. Show that

1. $\beta\mathbf{0} = \mathbf{0}$ for all $\beta \in \mathbb{R}$, and
2. if $\alpha\mathbf{x} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{x} = \mathbf{0}$.

1. $$\begin{aligned} \beta\bar{0} &= \beta(\bar{x} + (-\bar{x})) && \text{by A4} \\ &= \beta\bar{x} + \beta(-\bar{x}) && \text{by A5} \\ &= \beta\bar{x} + (-\beta\bar{x}) && \text{since } -\bar{x} = -1\cdot\bar{x} \text{ (from REs)} \\ &= \bar{0} && \text{by A4.} \end{aligned}$$ ◻

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2. Suppose $\alpha\bar{x} = \bar{0}$.

Then,

$$\bar{x} + \alpha\bar{x} = \bar{x}$$

$$\Rightarrow (1+\alpha)\bar{x} = 1\bar{x} \quad \text{by A6. and A8}$$

$$\Rightarrow \text{either } 1+\alpha = 1 \text{ or } \bar{x} = \bar{0}.$$

so $\alpha = 0$ ◻

3. Let $\mathbb{R}^+$ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by

$$\alpha \circ x = x^\alpha$$

for each $x \in \mathbb{R}^+$ and each $\alpha \in \mathbb{R}$. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y$$

for all $x, y \in \mathbb{R}^+$. Is $(\mathbb{R}^+, \oplus, \circ)$ a vector space? Prove or disprove your answer.

C1. $x, y \in \mathbb{R}^+$, $x \oplus y = x \cdot y \in \mathbb{R}^+$. ✓

C2. $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^+$, $x^\alpha > 0$ ✓.

A1. $x \oplus y = x \cdot y = y \cdot x = y \oplus x$ ✓

A2. $x \oplus (y \oplus z) = x \cdot (y \cdot z) = (x \cdot y) \cdot z = (x \oplus y) \oplus z$ ✓

A3. $\bar{0} = 1 \in \mathbb{R}^+ \rightarrow \bar{0} \oplus \bar{x} = 1 \cdot x = \bar{x}$. ✓

A4. for $x \in \mathbb{R}^+$, $(-\bar{x}) = \frac{1}{x} \rightarrow \bar{x} \oplus (-\bar{x}) = x \cdot \frac{1}{x} = 1 = \bar{0}$. ✓

A5. $\alpha \circ (x \oplus y) = (x \cdot y)^\alpha = x^\alpha y^\alpha = (\alpha \circ x) \oplus (\alpha \circ y)$. ✓

A6. $(\alpha + \beta) \circ x = x^{\alpha+\beta} = x^\alpha \cdot x^\beta = (\alpha \circ x) \oplus (\beta \circ x)$ ✓

A7. $(\alpha\beta) \circ x = x^{\alpha\beta} = (x^\beta)^\alpha = \alpha \circ (\beta \circ x)$
$= (x^\alpha)^\beta = \beta \circ (\alpha \circ x)$ ✓

A8. $1 \circ x = x^1 = x$ ✓

Since all 10 axioms are satisfied, this is a vector space.

4. Let $\mathbb{Z}$ denote the set of all integers with addition defined in the usual way, and define scalar multiplication, denoted $\circ$, by

$$\alpha \circ k = [\![\alpha]\!] \cdot k$$

for all $\alpha \in \mathbb{R}$ and all $k \in \mathbb{Z}$, where $[\![\ ]\!]$ is the greatest integer function. The space $(\mathbb{Z}, +, \circ)$ is not a vector space. Which axioms fail to hold, and why?

A6. Let $\alpha = \beta = 2.5$

Then $(\alpha + \beta) \circ k = 5 \circ k = [\![5]\!] k = 5k.$

but $\alpha \circ k + \beta \circ k = [\![2.5]\!] k + [\![2.5]\!] k = 2k + 2k = 4k$

and these are not equal.

Similarly,

A7. Let $\alpha = \beta = 1.9$

Then $(\alpha\beta) \circ k = [\![3.61]\!] k = 3k$

but $\alpha \circ (\beta \circ k) = [\![1.9]\!]^2 k = 1^2 k = k.$

and these are not equal.

5. Consider the vector spaces $\mathbb{P}_n$ and $\mathbb{R}^n$ with their usual additions and scalar multiplications, and consider the correspondence

$$p(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1} \leftrightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \mathbf{a}.$$

Show that

1. $\alpha p \leftrightarrow \alpha \mathbf{a}$ for every $\alpha \in \mathbb{R}$, and
2. $p + q \leftrightarrow \mathbf{a} + \mathbf{b}$.

1.) $\alpha p(x) = \alpha a_1 + \alpha a_2 x + \ldots + \alpha a_n x^{n-1} \leftrightarrow \begin{pmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_n \end{pmatrix} = \alpha \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \alpha \bar{a}.$

2. If $p(x) = a_1 + a_2 x + \ldots + a_n x^{n-1}$ and $q(x) = b_1 + b_2 x + \ldots + b_n x^{n-1}$,

then

$(p+q)(x) = (a_1+b_1) + (a_2+b_2)x + \ldots + (a_n+b_n)x^{n-1}$

$\leftrightarrow \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \bar{a} + \bar{b}.$