Name: Key
M511: Linear Algebra (Spring 2018)
Instructor: Justin Ryan
Good Problems 2: Sections 3.2, 3.3

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Determine the null space of the matrix.

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{pmatrix}$$

$$\left.\begin{matrix} R_2-2R_1 \to R_2 \\ R_3+R_1 \to R_3 \end{matrix}\right\} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & -1 & -3 & 0 \end{array}\right)$$

$$\left.\begin{matrix} R_2-R_3 \to R_3 \\ (-1)R_2 \to R_2 \end{matrix}\right\} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1+R_2 \to R_1 \Big\} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

lead free

$x_1 = -u-5t$
$x_2 = u$
$x_3 = -3t$
$x_4 = t$

so $$\boxed{N(A) = u\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}}$$

2. Determine whether the following are subspaces of $\mathbb{P}_4$. Clearly explain your answers.

1. The set of polynomials in $\mathbb{P}_4$ of even degree.
2. The set of polynomials of degree 3.
3. The set of all polynomials $p \in \mathbb{P}_4$ such that $p(0) = 0$.
4. The set of all polynomials in $\mathbb{P}_4$ having at least one real root.

2.1. No. Example: $p(x) = x^2 + x$, $q(x) = -x^2$, $(p+q)(x) = x$ is not of even degree.

2.2. No. Same reasoning as 2.1., or $\bar{0}(x) = 0x^3 + 0x^2 + 0x + 0$ is not of degree 3.

2.3. Yes. $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0.$, $(\alpha p)(0) = \alpha(p(0)) = \alpha \cdot 0 = 0.$

2.4. No. Example: $p(x) = x^3 + x^2$, $q(x) = 4 - x^3$, but $(p+q)(x) = x^2 + 4$ does not have a real root.

roots $= 0, -1$ real root $= \sqrt[3]{4}$

3. Which of the following are spanning sets for $\mathbb{P}_3$? Justify your answers.

1. $\{1, x^2, x^2 - 2\}$
2. $\{2, x^2, x, 2x + 3\}$
3. $\{x + 2, x + 1, x^2 - 1\}$
4. $\{x + 2, x^2 - 1\}$

3.1. NO: It's impossible to get a non-zero x-term.

3.2. Yes: $2c_1 + c_2x^2 + c_3x + c_4(2x+3) = ax^2+bx+c$

put $c_1 = \frac{c}{2}$ $c_2 = a$ $c_3 = b$ $c_4 = 0$.

3.3. Yes: Build the coefficient vectors: $\begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}$.

The determinant of the matrix is

$$\begin{vmatrix}0 & 0 & 1\\1 & 1 & 0\\2 & 1 & -1\end{vmatrix} = 0+0+1(1-2) = -1 \neq 0.$$

3.4. No: Not enough vectors in the set.

4. Prove that any finite set of vectors that contains the zero vector must be linearly dependent.

Consider the set $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-1}, \bar{0}\}$, and

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_{n-1}\bar{v}_{n-1} + c_n\bar{0} = \bar{0} \quad (*)$$

We can put $c_1 = c_2 = \ldots = c_{n-1} = 0$ and $c_n = 1$ and $(*)$ is satisfied.

Therefore $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-1}, \bar{0}\}$ are linearly dependent. $\square$

5. Prove that any nonempty subset of a linearly independent set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is also linearly independent.

Let $\{k_1, k_2, \ldots, k_m\}$ be a subset of $\{1, 2, 3, \ldots, n\}$ and suppose there exist $b_{k_1}, b_{k_2}, \ldots, b_{k_m}$, not all zero, such that

$$b_{k_1}\bar{v}_{k_1} + b_{k_2}\bar{v}_{k_2} + \cdots + b_{k_m}\bar{v}_{k_m} = \bar{0}$$

Now consider

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n = \bar{0}, \quad (*)$$

and put $c_i = b_{k_j}$ if $i = k_j$. Then equation $(*)$ is satisfied and not all c's are zero. This contradicts the assumption that $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ are linearly independent.

Hence the subset $\{\bar{v}_{k_1}, \bar{v}_{k_2}, \ldots, \bar{v}_{k_m}\}$ must be linearly independent. $\square$