Name: Key

M511: Linear Algebra (Spring 2018)
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Good Problems 3: Sections 3.4-3.5

WICHITA STATE UNIVERSITY

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Let S be the subspace of $\mathbb{P}_3$ consisting of all polynomials p such that $p(0) = 0$, and let T be the subspace of all polynomials q such that $q(1) = 0$. Find bases for S, T, and $S \cap T$.

$p(x) = ax^2 + bx + c$

$p(0) = 0 + 0 + c = 0 \Rightarrow c = 0$

So, $T = \text{span}\{x^2, x\}$

$q(x) = a(x-1)^2 + b(x-1) + c$

$q(1) = 0 + 0 + c = 0 \Rightarrow c = 0$

So, $S = \text{span}\{(x-1)^2, (x-1)\}$

$S \cap T$ must satisfy both $p(0) = 0$ and $p(1) = 0$.

So, $p(x) = ax^2 + bx \Rightarrow p(1) = a + b = 0 \Rightarrow b = -a$

then $p(x) = ax^2 - ax = a(x^2 - x)$

and $S \cap T = \text{span}\{x^2 - x\}$

or

$q(x) = a(x-1)^2 + b(x-1)$

$q(0) = a - b = 0 \Rightarrow a = b$

$q(x) = a(x-1)^2 + a(x-1)$

$= a((x-1)^2 + (x-1))$

$= a(x^2 - 2x + 1 + x - 1)$

$= a(x^2 - x)$

2. Show that if U and V are subspaces of $\mathbb{R}^n$ and $U \cap V = \{\mathbf{0}\}$, then

$$\dim(U + V) = \dim U + \dim V.$$

Let $U = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n\}$ be a basis of U and $V = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k\}$ be a basis of V.

Then $U+V = \text{span}\{U, V\} = \text{span}\{\bar{u}_1, \dots, \bar{u}_n, \bar{v}_1, \dots, \bar{v}_k\}$.

Suppose $\{\bar{u}_1, \dots, \bar{u}_n, \bar{v}_1, \dots, \bar{v}_k\}$ are linearly dependent.

Then there exists some $\bar{v} \in V$, $\bar{v} \neq \bar{0}$, s.t.

$\bar{v} = c_1\bar{u}_1 + c_2\bar{u}_2 + \dots + c_n\bar{u}_n$

But this contradicts $U \cap V = \{\bar{0}\}$.

therefore $\{\bar{u}_1, \dots, \bar{u}_n, \bar{v}_1, \dots, \bar{v}_k\}$ are linearly independent, hence form a basis of $U+V$.

And $\dim(U+V) = n + k = \dim U + \dim V$.

3. Given

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad S = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix},$$

find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that S will be the transition matrix from V to U.

$$S : V \to U : [\vec{x}]_V \mapsto S[\vec{x}]_V = [\vec{x}]_U$$

Then

$$\bar{u}_1 = S\bar{v}_1 = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 8+6 \\ 4+6 \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}$$

and $\bar{u}_2 = S\bar{v}_2 = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4+4 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$

4. Write the standard coordinates in $\mathbb{P}_7$ for the 6^{th} degree Taylor polynomial of $f(x) = \cos(x)$.

$$T_6(x) = \sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!} x^n = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{4!}x^4 + 0x^5 - \frac{1}{6!}x^6$$

$$\begin{array}{lll} & & \xrightarrow{x=0} \\ f^{(0)}(x) = f(x) & = \cos x & \quad 1 \\ f^{(1)}(x) = f'(x) & = -\sin x & \quad 0 \\ f^{(2)}(x) = f''(x) & = -\cos x & \quad -1 \\ f^{(3)}(x) & = \sin x & \quad 0 \\ f^{(4)}(x) & = \cos x & \quad 1 \\ f^{(5)}(x) & = -\sin x & \quad 0 \\ f^{(6)}(x) & = -\cos x & \quad -1 \end{array}$$

so

$$[T_6(x)]_E = \begin{pmatrix} 1 \\ 0 \\ -1/2 \\ 0 \\ 1/4! \\ 0 \\ -1/6! \end{pmatrix}_E$$

5. Let $U = \{x, 1\}$ and $V = \{2x-1, 2x+1\}$ be ordered bases for $\mathbb{P}_2$. Find the transition matrices $U \to V$ and $V \to U$.

$U = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad V = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

$V : V \to U$ is the transition matrix from V to U.

$V^{-1} : U \to V$ requires a little work.

$\det(V) = 2 + 2 = 4$

$$V^{-1} = \frac{1}{\det V} C_V^t = \frac{1}{4}\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/4 & -1/2 \\ 1/4 & 1/2 \end{pmatrix}$$