Name: Key

M511: Linear Algebra (Spring 2018)
Instructor: Justin Ryan
Good Problems 4: Sections 2.1, 3.5, and 3.6

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{pmatrix}.$$

a.) Find all minors of A.

b.) Find all cofactors of A.

c.) Compute the determinant of A.

d.) Compute A^{-1}, if it exists. If it doesn't exist, explain why.

a.) $M_{11} = \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}$ $M_{12} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ $M_{13} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$

$M_{21} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$ $M_{22} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$ $M_{23} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

$M_{31} = \begin{pmatrix} 2 & 4 \\ -2 & 3 \end{pmatrix}$ $M_{32} = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$ $M_{33} = \begin{pmatrix} 3 & 2 \\ 1 & -2 \end{pmatrix}$

b.) $A_{11} = -13$ $A_{12} = -(-8) = 8$ $A_{13} = 7$

$A_{21} = -(-16) = 16$ $A_{22} = -2$ $A_{23} = -5$

$A_{31} = 14$ $A_{32} = -5$ $A_{33} = -8$

c.) $\det(A) = 3(-13) + 2(8) + 4(7)$
$= -39 + 16 + 28$
$= -39 + 44 = 5$

d.)

$$A^{-1} = \frac{1}{5}\begin{pmatrix} -13 & 16 & 14 \\ 8 & -2 & -5 \\ 7 & -5 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} -13/5 & 16/5 & 14/5 \\ 8/5 & -2/5 & -1 \\ 7/5 & -1 & 8/5 \end{pmatrix}$$

2. Consider the subspace of $C^{\infty}(\mathbb{R})$, $S = \text{span}\{e^x, e^{-x}\}$.

a.) Show that every function $\varphi \in S$ is a solution of the second order differential equation $y'' - y = 0$.

b.) Show that $V = \left\{\sinh(x), \cosh(x)\right\}$ also forms a basis for S.

c.) Regarding $E = \left\{e^x, e^{-x}\right\}$ as the standard basis for S, find the transition matrices from $V \to E$ and $E \to V$.

a.) $\left.\begin{aligned} \varphi &= c_1 e^x + c_2 e^{-x} \\ \varphi' &= c_1 e^x - c_2 e^{-x} \\ \varphi'' &= c_1 e^x + c_2 e^{-x} \end{aligned}\right\}$ $\quad \varphi'' - \varphi = c_1 e^x + c_2 e^{-x} - (c_1 e^x + c_2 e^{-x})$

$= (c_1 e^x - c_1 e^x) + (c_2 e^{-x} - c_2 e^{-x})$

$= 0 - 0$

$= 0$. ✓

b.) $\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ and $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

$W(\sinh x, \cosh x) = \begin{vmatrix} \sinh x & \cosh x \\ \cosh x & \sinh x \end{vmatrix} = \sinh^2 x - \cosh^2 x = -1 \neq 0$ So they are linearly independent.

$\alpha \sinh x + \beta \cosh x = \frac{1}{2}\alpha e^x - \frac{1}{2}\alpha e^{-x} + \frac{1}{2}\beta e^x + \frac{1}{2}\beta e^{-x} = \frac{1}{2}(\alpha+\beta)e^x + \frac{1}{2}(\beta-\alpha)e^{-x}$

putting $\bar{x} = c_1 e^x + c_2 e^{-x}$, $\begin{aligned} 2c_1 &= \alpha + \beta \\ 2c_2 &= -\alpha + \beta \end{aligned}$ $\Rightarrow$ $\left.\begin{aligned} 2\beta &= 2c_1 + 2c_2 \quad \text{or} \quad \beta = c_1 + c_2 \\ -2\alpha &= 2c_2 - 2c_1 \quad \text{or} \quad \alpha = c_1 - c_2 \end{aligned}\right\}$ so they span S.

Hence, $\{\sinh x, \cosh x\}$ form a basis for S.

c.) $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$

V is the transition matrix from $V \to E$.

$V^{-1} = \frac{1}{1/2}\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the transition matrix from $E \to V$.

3. Consider the vectors in $\mathbb{R}^2$,

$$\mathbf{u}_1 = \begin{pmatrix}1\\2\end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix}2\\3\end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix}2\\6\end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix}1\\4\end{pmatrix}.$$

Letting $U = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $V = \{\mathbf{v}_1, \mathbf{v}_2\}$, find the transition matrices $U \to V$ and $V \to U$.

$U = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}$, $\quad V = \begin{pmatrix}2 & 1\\6 & 4\end{pmatrix}$, $\quad S = V^{-1}U : U \to V$

$\det(V) = 8 - 6 = 2$ so $V^{-1} = \frac{1}{2}\begin{pmatrix}4 & -1\\-6 & 2\end{pmatrix}$ and

$$S = \frac{1}{2}\begin{pmatrix}4 & -1\\-6 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 2\\-2 & -6\end{pmatrix} = \begin{pmatrix}1 & 1\\-1 & -3\end{pmatrix}$$

Therefore $S^{-1} : V \to U$ is given by

$$S^{-1} = \frac{1}{-2}\begin{pmatrix}-3 & -1\\1 & 1\end{pmatrix} = \begin{pmatrix}3/2 & 1/2\\-1/2 & -1/2\end{pmatrix}$$

4. Consider the matrix

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix}.$$

Find bases for the row, column, and null spaces of A.

Begin with REF:

$$\begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix} \xrightarrow[\substack{R_1 - R_3 \to R_3 \\ R_1 + 3R_2 \to R_1 \\ R_1 \leftrightarrow R_2}]{} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & -7 & -1 & 2 \end{pmatrix} \xrightarrow[\substack{R_2 + R_3 \to R_3 \\ \frac{1}{7}R_2 \to R_2 \\ -R_3 \to R_3}]{} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Therefore,

$$\text{Row}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ or } \text{Row}(A) = \text{span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 8 \\ 4 \\ 2 \end{pmatrix} \right\}$$

$$\text{Col}(A) = \text{span}\left\{ \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\} = \mathbb{R}^3$$

so for this problem any basis of $\mathbb{R}^3$ is acceptable. (In general you need to use the columns of A as the basis though.)

Null(A):

$$x_4 = t, \quad x_3 = 0, \quad x_2 = \tfrac{2}{7}t, \quad x_1 = 2t + 0 - 2\left(\tfrac{2}{7}\right)t = \tfrac{10}{7}t$$

Therefore $\text{Null}(A) = \text{span}\left\{ \begin{pmatrix} 10 \\ 2 \\ 0 \\ 7 \end{pmatrix} \right\}$