Name: Key

M511: Linear Algebra (Spring 2018)
Instructor: Justin Ryan
Good Problems 5: Sections 3.5, 3.6, 4.1 and 4.2

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Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Consider the transformation $L : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $L(p)(x) = x \cdot p'(x) + p(0)$.

a.) Prove that L is linear.

$\underline{L1}$. $L(p+q) = x\cdot(p+q)'(x) + (p+q)(0)$

$= x\cdot(p'(x)+q'(x)) + (p(0)+q(0))$

$= x\cdot p'(x) + p(0) + x\cdot q'(x) + q(0) = L(p) + L(q)$. ✓

$\underline{L2}$. $L(\alpha p) = x\cdot(\alpha p)'(x) + (\alpha p)(0)$

$= x(\alpha\, p'(x)) + \alpha\,(p(0))$

$= \alpha\,(x\cdot p'(x) + p(0)) = \alpha L(p)$. ✓

b.) Find the matrix representing L with respect to the ordered basis $\{1, (x-1), (x-1)^2\} =: V$ ✓

$E = \{1, x, x^2\}$:

$L(1) = x\cdot 0 + 1 = 1$

$L(x) = x\cdot 1 + 0 = x$

$L(x^2) = x\cdot 2x + 0 = 2x^2$

So the matrix representing L wrt E is

$$A_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The matrix we want must obey the diagram:

$$\begin{tikzcd} V \arrow[r, "A_V"] \arrow[d, dashed, "V"'] & V \\ E \arrow[r, dashed, "A_E"'] & E \arrow[u, dashed, "V^{-1}"'] \end{tikzcd}$$

$[\vec v_1]_V = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$[\vec v_2]_V = -1 + x = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$[\vec v_3]_V = 1 - 2x + x^2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

So,

$$V = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$
$V \qquad I$

$\to$ $R_1 - R_3 \to R_1$
$R_2 + 2R_3 \to R_2$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$R_1 + R_2 \to R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$
$I \qquad V^{-1}$

Now, $A_V = V^{-1} A_E V$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

So the matrix representing L wrt the ordered basis V 13

$$\boxed{A_V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}}$$

2. Let L be a linear transformation $L:\mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$L\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} \quad \text{and} \quad L\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-3\\-1\end{pmatrix}.$$

a.) Write the matrices representing L with respect to the basis $U = \left\{\begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$, and with respect to the standard basis of $\mathbb{R}^2$.

b.) Compute $L\begin{pmatrix}4\\4\end{pmatrix}$, and write your answer in coordinates with respect to the basis $V = \left\{\begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}-3\\-1\end{pmatrix}\right\}$.

a.) $[L(\bar{u}_1)]_E = \begin{pmatrix}2\\4\end{pmatrix}$, $[L(\bar{u}_2)]_E = \begin{pmatrix}-3\\-1\end{pmatrix}$, $U: U \to E$, $U^{-1}: E \to U$

$U = \begin{pmatrix}-2 & 1\\1 & -1\end{pmatrix}$, $U^{-1} = \frac{1}{1}\begin{pmatrix}-1 & -1\\-1 & -2\end{pmatrix}$

$[L(\bar{u}_1)]_U = U^{-1}[L(\bar{u}_1)]_E = \begin{pmatrix}-1 & -1\\-1 & -2\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}-6\\-10\end{pmatrix}_U$

$[L(\bar{u}_2)]_U = U^{-1}[L(\bar{u}_1)]_E = \begin{pmatrix}-1 & -1\\-1 & -2\end{pmatrix}\begin{pmatrix}-3\\-1\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}_U$, so $\boxed{A_U = \begin{pmatrix}-6 & 4\\-10 & 5\end{pmatrix}}$

$$\begin{tikzcd} U \arrow[r, "A_U"] & U \arrow[d, dashed, "U"] \\ E \arrow[u, dashed, "U^{-1}"] \arrow[r, "A_E"'] & E \end{tikzcd}$$

$A_E = U A_U U^{-1} = \begin{pmatrix}-2 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}-6 & 4\\-10 & 5\end{pmatrix}\begin{pmatrix}-1 & -1\\-1 & -2\end{pmatrix} = \begin{pmatrix}-2 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}2 & -2\\5 & 0\end{pmatrix} = \boxed{\begin{pmatrix}1 & 4\\-3 & -2\end{pmatrix} = A_E}$

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b.) $\begin{pmatrix}4\\4\end{pmatrix}$ is in E-coordinates, so $[L\begin{pmatrix}4\\4\end{pmatrix}]_V = V^{-1}A_E\begin{pmatrix}4\\4\end{pmatrix}$.

$V = \begin{pmatrix}2 & -3\\4 & -1\end{pmatrix}$ so $V^{-1} = \frac{1}{-2+12}\begin{pmatrix}-1 & 3\\-4 & 2\end{pmatrix} = \frac{1}{10}\begin{pmatrix}-1 & 3\\-4 & 2\end{pmatrix}$

$[L\begin{pmatrix}4\\4\end{pmatrix}]_V = \frac{1}{10}\begin{pmatrix}-1 & 3\\-4 & 2\end{pmatrix}\begin{pmatrix}1 & 4\\-3 & -2\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix} = \frac{1}{10}\begin{pmatrix}-1 & 3\\-4 & 2\end{pmatrix}\begin{pmatrix}20\\-20\end{pmatrix} = \begin{pmatrix}-1 & 3\\4 & 2\end{pmatrix}\begin{pmatrix}2\\-2\end{pmatrix} = \boxed{\begin{pmatrix}-8\\4\end{pmatrix}_V}$

3. Let $A \in \mathbb{R}^{3\times 5}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_5$. Further suppose that $\mathbf{a}_1$ and $\mathbf{a}_3$ are linearly independent, $\mathbf{a}_2 = 2\mathbf{a}_1$, $\mathbf{a}_4 = \mathbf{a}_1 + \mathbf{a}_3$, and $\mathbf{a}_5 = \mathbf{a}_4 - \mathbf{a}_2$.

a.) What is the reduced row echelon form (RREF) of A?

$\{\bar{a}_1, \bar{a}_3\}$ are l.i., hence form a basis for col(A).

$\bar{a}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\bar{a}_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\bar{a}_2 = \begin{pmatrix}2\\0\\0\end{pmatrix}$, $\bar{a}_4 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\bar{a}_5 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

So $\boxed{\text{rref}(A) = \begin{pmatrix}1 & 2 & 0 & 1 & -1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}}$

b.) What is the column space of A?

$\text{Col}(A) = \text{span}\{\bar{a}_1, \bar{a}_3\}$

(This could be done before part a.)

<u>Q</u>. What is the row space of A?

4. Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ be a basis of $\mathbb{R}^4$ and $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of $\mathbb{R}^3$. Suppose $L: \mathbb{R}^4 \to \mathbb{R}^3$ is the linear transformation defined by

$$L(\mathbf{x}) = x_4\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 - x_3)\mathbf{b}_3,$$

where $\mathbf{x} = \left[(x_1, x_2, x_3, x_4)^T\right]_A$ is given in A-coordinates. Write the matrix representing L with respect to the bases A and B.

$L\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = L(\bar{a}_1) = \bar{b}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$L\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = L(\bar{a}_2) = \bar{b}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$L\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = L(\bar{a}_3) = -\bar{b}_3 = \begin{pmatrix}0\\0\\-1\end{pmatrix}$

$L\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = L(\bar{a}_4) = \bar{b}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

So, $\boxed{E_L = \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0\end{pmatrix}}$