GP8

1. $A = \begin{pmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & -14 \\ 0 & -5 \\ 0 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$

$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \\ 4 \end{pmatrix} \right\}$

$\mathrm{Null}(A) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

Fundamental Theorem the implies:

$\mathrm{Col}(A^T) = \mathbb{R}^2 = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

Null(AT) can be computed as $(\mathrm{Col}(A))^\perp$ or directly from the definition.

$A^T = \begin{pmatrix} 4 & 1 & 2 & 3 \\ -2 & 3 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 4 & 1 & 2 & 3 \\ 0 & 7 & 4 & 11 \end{pmatrix} \to \begin{pmatrix} 28 & 0 & 10 & 10 \\ 0 & 7 & 4 & 11 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5/14 & 5/14 \\ 0 & 1 & 4/7 & 11/7 \end{pmatrix}$

So $\mathrm{Null}(A^T) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ -15/7 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -5/7 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

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2. 1. $\bar{x} \in \mathrm{Null}(A^TA) \Rightarrow A^TA\bar{x} = \bar{0}$.

$A\bar{x} \in \mathrm{Col}(A)$ by definition: $\mathrm{Col}(A) = \mathrm{Range}(A)$.

$A\bar{x} \in \mathrm{Null}(A^T)$ since $A^T(A\bar{x}) = \bar{0}$. (again, by definition)

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2. ($\Leftarrow$) if $\bar{x} \in \mathrm{Null}(A)$, then $A\bar{x} = \bar{0}$ and $A^TA\bar{x} = A^T\bar{0} = \bar{0}$, so $\bar{x} \in \mathrm{Null}(A^TA)$.

($\Rightarrow$) if $\bar{x} \in \mathrm{Null}(A^TA)$, then $A\bar{x}$ is in both $\mathrm{Col}(A)$ and $\mathrm{Null}(A^T)$. These are orthogonal spaces by the Fundamental Theorem, so they only coincide for $A\bar{x} = \bar{0}$. Hence $\bar{x} \in \mathrm{Null}(A)$.

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3. $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = \dim(\mathrm{Col}(A^T))$. Meanwhile, $\mathrm{Col}(A^TA) = \mathrm{Col}(A^T)$, so $\dim(\mathrm{Col}(A^TA)) = \dim(\mathrm{Col}(A))$, and $rk(A^TA) = rk(A)$.

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4. $A \in \mathbb{R}^{m\times n}$, $n \leq m$. If $A$ has l.i. columns, then $rk(A) = n$.

$A^TA \in \mathbb{R}^{n\times n}$ also has rank $n$ by above, whence the columns of $A^TA$ are linearly independent. Since $A^TA$ is a square matrix, $\det(A^TA)$ is defined. Since $rk(A^TA) = n$, $\det(A^TA) \neq 0$. Thus $A^TA$ is nonsingular.

3.

| $x$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|
| $y$ | $0$ | $1$ | $3$ | $9$ |

$$y = mx + b \Rightarrow \bar{y} = A\bar{m}, \quad A = \begin{array}{cc} m & b \\ \left(\begin{array}{c} \\ \\ \end{array}\right. & \left.\begin{array}{c} \\ \\ \end{array}\right) \end{array} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad \bar{y} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}, \quad (A^TA)^{-1} = \frac{1}{20}\begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\hat{m} = (A^TA)^{-1}A^T\bar{y} = \frac{1}{10}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 21 \\ 13 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 29 \\ 18 \end{pmatrix}$$

so the least squares best fit line for this data is:

$$\boxed{y = \frac{29}{10}x + \frac{9}{5}}$$

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4. $(x-h)^2 + (y-k)^2 = r^2$ is the general equation of a circle. Our job is to use the data to find $h, k, r$. To make this a linear system, we expand:

$$x^2 - 2xh + h^2 + y^2 - 2yk + k^2 = r^2$$

$$x^2 + y^2 = 2xh + 2yk + \underbrace{r^2 - h^2 - k^2}_{u}$$

$$\bar{b} = A\hat{x} \quad \text{w/} \quad \hat{x} = \begin{pmatrix} 2h \\ 2k \\ u \end{pmatrix}$$

Then, $A = \begin{array}{ccc} \bar{x} & \bar{y} & 1 \\ \left(\begin{array}{c} \\ \\ \end{array}\right. & & \left.\begin{array}{c} \\ \\ \end{array}\right) \end{array}$

or $A = \begin{pmatrix} -1 & -2 & 1 \\ 0 & 2.4 & 1 \\ 1.1 & -4 & 1 \\ 2.4 & -1.6 & 1 \end{pmatrix}$

and $\bar{b} = \begin{pmatrix} 5 \\ 5.76 \\ 17.21 \\ 8.32 \end{pmatrix}$

Now solve $\hat{x} = (A^TA)^{-1}A^T\bar{b}$ using a computer to get, rounding,

$$\hat{x} = \begin{pmatrix} 1.15 \\ -1.29 \\ 6.68 \end{pmatrix}$$

so $2h = 1.15$, $2k = -1.29$, and

$r^2 - h^2 - k^2 = 6.68$.

We end up w/

$h = 0.575$, $k = -0.645$, and

$r^2 = 7.427$. The circle is:

$$\boxed{(x - 0.575)^2 + (y + 0.645)^2 = 7.427}$$