Name:__________________________________
M511: Linear Algebra (Spring 2018)
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Good Problems 9: Section 5.4

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Consider the inner product $\langle\ ,\ \rangle$ on $V = \mathbb{P}_4$.

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Find the matrix G representing $\langle\ ,\ \rangle$ with respect to the standard basis.

Use the matrix G to compute $\langle p, q\rangle$ where $p(x) = x^3 - 2x^2 + x$ and $q(x) = x^2 - 9x + 3$.

2. Let V be a vector space with inner product $\langle\ ,\ \rangle$. The *length* or *magnitude* of any vector $\mathbf{x} \in V$ is defined to be

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Find the lengths of the vectors $p(x) = x^3 - 2x^2 + x$ and $q(x) = x^2 - 9x + 3$ with respect to each of the following inner products on $\mathbb{P}_4$.

$$\begin{aligned} \langle p, q \rangle_1 &= \int_{-1}^{1} p(x)q(x)\,dx, \quad \text{and} \\ \langle p, q \rangle_2 &= p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3) + p(x_4)q(x_4), \end{aligned}$$

where $\mathbf{x} = (-2, -1, 1, 2)^T$.

3. Let $\mathbf{x}, \mathbf{y} \in V$, and let $\langle\ ,\ \rangle$ be an inner product on V. The *vector projection of* $\mathbf{x}$ *onto* $\mathbf{y}$ is defined to be

$$\text{proj}_{\mathbf{y}}\mathbf{x} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{y}\|} \frac{\mathbf{y}}{\|\mathbf{y}\|}.$$

Compute $\text{proj}_p q$ with respect to both of the inner products defined in problem **2**.

4. Prove the *General Cauchy-Schwarz-Bunyachevsky Inequality*:

Let $(V, \langle\ ,\ \rangle)$ be an inner product space. Then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|,$$

and equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Hint: Take $\text{proj}_{\mathbf{y}}\mathbf{x}$ and apply the Pythagorean Theorem.

From the proof of *CSB*, define the angle between $\mathbf{x}$ and $\mathbf{y}$ by

$$\theta(\mathbf{x}, \mathbf{y}) = \cos^{-1}\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right).$$

Check that this is well-defined.