GP9

1. $B = \{1, x, x^2, x^3\}$

$\langle 1,1\rangle = \int_{-1}^{1} 1\,dx = 2$ $\quad\langle 1,x\rangle = \int_{-1}^{1} x\,dx = 0$ $\quad\langle x,x^2\rangle = \int_{-1}^{1} x^3\,dx = 0$ $\quad\langle x^2,x^3\rangle = \int_{-1}^{1} x^5\,dx = 0$

$\langle x,x\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$ $\quad\langle 1,x^2\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$ $\quad\langle x,x^3\rangle = \int_{-1}^{1} x^4\,dx = \frac{2}{5}$

$\langle 1,x^3\rangle = \int_{-1}^{1} x^3\,dx = 0$

$\langle x^2,x^2\rangle = \int_{-1}^{1} x^4\,dx = \frac{2}{5}$

$\langle x^3,x^3\rangle = \int_{-1}^{1} x^6\,dx = \frac{2}{7}$

a.) So,
$$\boxed{G = \begin{pmatrix} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{pmatrix}}$$

b) $p(x) = x^3 - 2x^2 + x \rightarrow [p] = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}$, $q(x) = x^2 - 9x + 3 \rightarrow [q] = \begin{pmatrix} 3 \\ -9 \\ 1 \\ 0 \end{pmatrix}$

$$\langle p,q\rangle = [p]^T G [q] = (0, 1, -2, 1)\begin{pmatrix} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{pmatrix}\begin{pmatrix} 3 \\ -9 \\ 1 \\ 0 \end{pmatrix} = \frac{-72}{5}$$

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2. These inner products just need to be computed by brute force, using the definitions or matrix rep's. You should get:

wrt $\langle,\rangle_1$:	wrt $\langle,\rangle_2$:
$\|p\|_1 = \sqrt{\frac{352}{105}}$	$\|p\|_2 = \sqrt{344}$
$\|q\|_1 = \sqrt{\frac{382}{5}}$	$\|q\|_2 = \sqrt{940}$

3.) Again, this is just a brute force computation using results from the previous questions. You should get:

WRT $\langle , \rangle_1$: $\text{proj}_{\underline{1}\, p}\, q = -\frac{189}{44}x^3 + \frac{189}{22}x^2 - \frac{189}{44}x$

and

WRT $\langle , \rangle_2$: $\text{proj}_{2\, p}\, q = \frac{-131}{86}x^3 + \frac{131}{43}x^2 - \frac{131}{86}x$

4.) If $\bar{y} = \bar{0}$, then $\langle \bar{x}, \bar{y} \rangle = \bar{0}$ and $\|\bar{y}\| = 0$, so the inequality is trivial.

If $\bar{y} \neq 0$, let $\bar{p} = \frac{\langle \bar{x}, \bar{y} \rangle}{\langle \bar{y}, \bar{y} \rangle}\bar{y} = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{y}\|}\frac{\bar{y}}{\|\bar{y}\|}$.

Then $\bar{z} = \bar{x} - \bar{p}$ and $\bar{p}$ are orthogonal, and

$$\|\bar{p}\|^2 + \|\bar{z}\|^2 = \|\bar{x}\|^2$$

$$\Rightarrow \|\bar{p}\|^2 = \|\bar{x}\|^2 - \|\bar{z}\|^2$$

$$\left(\frac{|\langle \bar{x}, \bar{y} \rangle|}{\|\bar{y}\|}\right)^2 = \|\bar{x}\|^2 - \|\bar{z}\|^2$$

$$|\langle \bar{x}, \bar{y} \rangle|^2 = \|\bar{x}\|^2\|\bar{y}\|^2 - \|\bar{z}\|^2\|\bar{y}\|^2$$

$$|\langle \bar{x}, \bar{y} \rangle|^2 \leq \|\bar{x}\|^2\|\bar{y}\|^2$$

$$\Rightarrow |\langle \bar{x}, \bar{y} \rangle| \leq \|\bar{x}\|\|\bar{y}\|$$

Equality holds only if $\bar{z} = \bar{0}$, whence $\bar{x} = \bar{p}$ and $\bar{x}$ and $\bar{y}$ are linearly dependent. □

b.) From this, we deduce that $-1 \leq \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\|\|\bar{y}\|} \leq 1$ for any inner product, and define $\theta = \arccos\left(\frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\|\|\bar{y}\|}\right)$.