GP11

1. $\bar{u}_1 = \dfrac{\bar{x}_1}{\|\bar{x}_1\|}$, $\|\bar{x}_1\| = \sqrt{16+4+4+1} = 5 \Rightarrow \boxed{\bar{u}_1 = \begin{pmatrix} 4/5 \\ 2/5 \\ 2/5 \\ 1/5 \end{pmatrix}}$

$$\bar{p}_2 = \langle \bar{u}_1, \bar{x}_2 \rangle \bar{u}_1 = \left(\left(4/5, 2/5, 2/5, 1/5\right) \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right) \frac{1}{5} \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \left(\frac{1}{5} \cdot (8+0+0+2) \right) \left(\frac{1}{5} \right) \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\bar{r}_2 = \bar{x}_2 - \bar{p}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} - \frac{2}{5} \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 \\ -4 \\ -4 \\ 8 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 1 \\ -2 \\ -2 \\ 4 \end{pmatrix}$$

so $\bar{u}_2 = \dfrac{\bar{r}_2}{\|\bar{r}_2\|} = \boxed{\frac{1}{5} \begin{pmatrix} 1 \\ -2 \\ -2 \\ 4 \end{pmatrix} = \bar{u}_2}$

$$\bar{p}_3 = \langle \bar{u}_1, \bar{x}_3 \rangle \bar{u}_1 + \langle \bar{u}_2, \bar{x}_3 \rangle \bar{u}_2 = \left(\frac{1}{5} (4,2,2,1) \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right) \frac{1}{5} \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix} + \left(\frac{1}{5} (1,-2,-2,4) \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right) \frac{1}{5} \begin{pmatrix} 1 \\ -2 \\ -2 \\ 4 \end{pmatrix}$$

$$= \left(\frac{1}{5} \left(4+2-2+1 \right) \right) \frac{1}{5} \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix} + \left(\frac{1}{5} (1-2+2+4) \right) \frac{1}{5} \begin{pmatrix} 1 \\ -2 \\ -2 \\ 4 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 5 \\ 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\bar{r}_3 = \bar{x}_3 - \bar{p}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$\bar{u}_3 = \dfrac{\bar{r}_3}{\|\bar{r}_3\|} = \boxed{\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \bar{u}_3}$

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2. a.) $\bar{u}_1 = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\bar{p}_2 = \left(\frac{1}{3}(2,1,2) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{9}(5) \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{5}{9} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\bar{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{9} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$, $\bar{u}_2 = \frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}} \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$

so, $\boxed{Q = \frac{1}{3\sqrt{2}} \begin{pmatrix} 2\sqrt{2} & -1 \\ \sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{pmatrix}}$

2.b.) $R = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ $\quad QR = A \Rightarrow \frac{1}{3\sqrt{2}} \begin{pmatrix} 2\sqrt{2} & -1 \\ \sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$

$$\Rightarrow \left.\begin{array}{l} \frac{2}{3}a = 2 \\ \frac{1}{3}a = 1 \\ \frac{2}{3}a = 2 \end{array}\right\} a = 3. \quad \text{and,} \left\{\begin{array}{l} \frac{2}{3}b - \frac{1}{3\sqrt{2}}c = 1 \\ \frac{1}{3}b + \frac{4}{3\sqrt{2}}c = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} (2\sqrt{2}\,b - c = 3\sqrt{2})4 \\ \sqrt{2}\,b + 4c = 3\sqrt{2} \end{array}\right.$$

$$9\sqrt{2}\,b = 15\sqrt{2} \rightarrow b = \frac{15}{9} = \frac{5}{3}.$$

$$c = 2\sqrt{2}\left(\frac{5}{3}\right) - 3\sqrt{2} = \left(\frac{10}{3} - \frac{9}{3}\right)\sqrt{2} = \frac{\sqrt{2}}{3}$$

So, $\boxed{R = \begin{pmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{pmatrix}}$

2.c.) $A\bar{x} = \bar{b} \rightarrow QR\bar{x} = \bar{b} \rightarrow \underbrace{Q^T Q}_{I} R\bar{x} = Q^T\bar{b} \rightarrow R\bar{x} = Q^T\bar{b}$

$$\rightarrow \hat{x} = R^{-1}Q^T\bar{b}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}/3 & -5/3 \\ 0 & 3 \end{pmatrix} \cdot \frac{1}{3\sqrt{2}} \begin{pmatrix} 2\sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ -1 & 4 & -1 \end{pmatrix} \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$$

$$\boxed{\hat{x} = \begin{pmatrix} 9\sqrt{2} \\ -3\sqrt{2} \end{pmatrix}}$$

3. Start w/ $E = \{1, x, x^2, x^3, x^4\}$

$\bar{u}_1 = P_0 = \frac{1}{\|1\|}$, $\|1\| = \sqrt{\int_{-1}^{1} 1\, dx} = 2$, so $\boxed{P_0 = \frac{1}{\sqrt{2}}}$.

$\bar{P}_2 = \langle \bar{u}_1, \bar{x}_2 \rangle \bar{u}_1 = \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} x \, dx \right) \frac{1}{\sqrt{2}} = 0.$

$\bar{r}_2 = \bar{x}_2 = x$

$\|\bar{r}_2\| = \sqrt{\int_{-1}^{1} x^2 dx} = \sqrt{\frac{2}{3}} \Rightarrow \boxed{P_1 = \frac{\sqrt{3}}{\sqrt{2}} x.}$

$\bar{P}_3 = \langle \bar{u}_1, \bar{x}_3 \rangle \bar{u}_1 + \langle \bar{u}_2, \bar{x}_3 \rangle \bar{u}_2 = \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 dx \right) \cdot \frac{1}{\sqrt{2}} + \left(\int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}} x^3 \, dx \right) \cdot \frac{\sqrt{3}}{\sqrt{2}} x = \frac{1}{3} + 0x = \frac{1}{3}$

$\bar{r}_3 = \bar{x}_3 - \bar{P}_3 = x^2 - \frac{1}{3}$

$\|\bar{r}_3\| = \sqrt{\int_{-1}^{1} (x^2 - 1/3)^2 \, dx} = \sqrt{\int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9} \, dx} = \sqrt{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}} = \sqrt{\frac{2}{5} - \frac{2}{9}} = \sqrt{\frac{8}{45}} = \frac{2\sqrt{2}}{3\sqrt{5}}$

so $\bar{u}_3 = \boxed{\frac{3\sqrt{5}}{2\sqrt{2}} x^2 - \frac{\sqrt{5}}{2\sqrt{2}} = P_2}$

... continue this process to get:

$\boxed{P_3 = \frac{5\sqrt{7}}{2\sqrt{2}} x^3 - \frac{3\sqrt{7}}{2\sqrt{2}} x}$ and

$\boxed{P_4 = \frac{105}{8\sqrt{2}} x^4 - \frac{45}{4\sqrt{2}} x^2 + \frac{9}{8\sqrt{2}}}$

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4. Don't do #4. Not a well-posed problem. Sorry "