Name: Key
M511: Linear Algebra (Spring 2018)
Instructor: Justin Ryan
Good Problems 13: Sections 6.1 and 6.2

WICHITA STATE UNIVERSITY

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Find the eigenvalues and bases for the eigenspaces of the matrix.

$$A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) - 6$$
$$= (\lambda-3)(\lambda+2) - 6$$
$$= \lambda^2 - \lambda - 6 - 6$$
$$= \lambda^2 - \lambda - 12$$
$$= (\lambda-4)(\lambda+3) = 0$$

Eigenvalues:
$$\boxed{\lambda_1 = 4, \quad \lambda_2 = -3}$$

$$(A-4I) = \left(\begin{array}{cc|c} 3-4 & 2 & 0 \\ 3 & -2-4 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 2 & 0 \\ \cancel{3} & \cancel{-6} & \cancel{0} \end{array}\right) \Rightarrow \bar{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$(A+3I) = \left(\begin{array}{cc|c} 3+3 & 2 & 0 \\ 3 & -2+3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} \cancel{6} & \cancel{2} & \cancel{0} \\ 3 & 1 & 0 \end{array}\right) \Rightarrow \bar{x}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Corresponding eigenvectors:
$$\boxed{\bar{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \bar{x}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$$

2. Find the eigenvalues and bases for the eigenspaces of the matrix.

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(-2-\lambda)(2-\lambda)+3\right] + 3\left[(2-\lambda)-1\right] + 1\left[-3+(\lambda+2)\right] =$$

$$= (2-\lambda)\left[(\lambda-2)(\lambda+2)+3\right] - 3\left[(\lambda-2)+1\right] + \left[(\lambda+2)-3\right]$$

$$= -(\lambda-2)(\lambda^2-4) - 3(\lambda-2) - 3(\lambda-2) - 3 + (\lambda+2) - 3$$

$$= (\lambda-2)\left[-(\lambda^2-4)-3-3\right] + (\lambda+2) - 6$$

$$= (\lambda-2)\left[-\lambda^2-2\right] + \lambda - 4$$

$$= -\lambda^3 + 2\lambda^2 - 2\lambda + 4 + \lambda - 4$$

$$= -\lambda^3 + 2\lambda^2 - \lambda = 0$$

$$-\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$-\lambda(\lambda-1)^2 = 0$$

Eigenvalues:

$$\boxed{\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1}$$

The corresponding eigenvectors are:

$$\boxed{\bar{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \ \bar{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ \bar{x}_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}}$$

$(A - 0I) =$ (Red pen died ☺)

$$\left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right) \xrightarrow[R_1-2R_3 \to R_3]{R_1-2R_2 \to R_2} \left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \xrightarrow{R_3-3R_2 \to R_3} \left(\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = 1 \Rightarrow x_2 = 1 \Rightarrow x_1 = 3 - 1 = 2.$ So $\bar{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$(A - I) = \left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

~~[illegible]~~

$x_3 = \alpha, \ x_2 = \beta \Rightarrow x_1 = 3\beta - \alpha$

$$\bar{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{x}_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

3. Consider the matrices T, S, and $A = S^{-1}TS$. Show that T and A have the same eigenvalues. What can you say about the corresponding eigenvectors?

$$T = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \quad S = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$S^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \qquad \text{so,} \quad A = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$= \frac{1}{\cancel{4}\,1}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 13 & 8 \\ 9 & 6 \end{pmatrix}$$

$$= \frac{1}{\cancel{4}\,1}\boxed{\begin{pmatrix} -1 & -2 \\ 6 & 6 \end{pmatrix}} = A \quad \cancel{= \begin{pmatrix} -1/4 & -1/2 \\ 6/4 & 6/4 \end{pmatrix}}$$

$$|A - \lambda I| = \begin{pmatrix} -1/4 - \lambda & -1/2 \\ 6/4 & 6/4 - \lambda \end{pmatrix} = (\lambda + 1/4)(\lambda - 6/4) + \frac{6}{8}$$

$$= \lambda^2 - \frac{5}{4}\lambda - \frac{6}{16} + \frac{6}{8}$$

$$= \lambda^2 - \frac{5}{4}\lambda + \frac{6}{16}$$

$$= \left(\lambda - \frac{3}{4}\right)\left(\lambda - \frac{2}{4}\right)$$

E'values of A:

$$\Rightarrow \boxed{\lambda_1 = \frac{3}{\cancel{4}}, \ \lambda_2 = \frac{2}{\cancel{4}}.} \qquad \lambda_2 = 3, \ \lambda_1 = 2$$

$$|T - \lambda I| = \begin{pmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = (\lambda - 2)(\lambda - 3) = 0$$

E'values of T:

$$\boxed{\lambda_1 = 2, \ \lambda_2 = 3.}$$

They are the same!

(well, they should have been...)

E'vectors of T:

$$\lambda_1 = 2: \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \Rightarrow \bar{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_2 = 3: \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

E'vectors of T are: $\boxed{\bar{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \bar{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

E'vectors of A are $S\bar{x}_1$ and $S\bar{x}_2$!

$$\boxed{S\bar{x}_1 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}} \quad \text{and} \quad \boxed{S\bar{x}_2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}}$$

4. Use the power series for e^x centered at $x_0 = 0$ to deduce that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

What does $e^{i\pi}$ equal?

Recall:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \cdots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2k)!} x^{2k} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + - \cdots$$

Then,

$$e^{i\theta} = 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \frac{1}{6!}(i\theta)^6 + \cdots$$

$$= 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 - \frac{1}{6!}\theta^6 - i\frac{1}{7!}\theta^7 + + - - \cdots$$

$$= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + - \cdots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + - \cdots\right)$$

$$= \cos\theta + i\sin\theta. \quad \blacksquare$$

Plugging in $\theta = \pi$:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1$$

So $\boxed{e^{i\pi} + 1 = 0}$:)