Name: Key

M511: Linear Algebra
Summer 2018
Midterm Exam: Chapters 1–3 (part I)

Instructions. *Complete all problems below, showing enough work. Read carefully and follow all instructions. You may not use any notes or electronic devices. All you need is a pencil and your brain.*

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1. True/False [20 points] *Neatly write* **T** *on the line if the statement is always true, and* **F** *otherwise [1 point each]. In the space provided below the statement, give sufficient explanation of your answer [3 point each].*

T **1.a.** Every homogeneous linear system is consistent.

$\bar{x} = \bar{0}$ is always a solution.

F **1.b.** Let $A \in \mathbb{R}^{n\times n}$. If A is nonsingular and $A = A^{-1}$, then either $A = I$ or $A = -I$.

e.g. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $AA = I$ so $A^{-1} = A$, but $A \neq I, -I$,

In general, any type I.

F **1.c.** Let $A \in R^{n\times n}$ and $k \in \mathbb{R}$. Then $\det(k \cdot A) = k \cdot \det(A)$.

$\det(kA) = k^n \det(A)$.

T **1.d.** If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ are vectors in a vector space V and $\text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} = \text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$, then $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ are linearly dependent.

$\bar{x}_k = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_{k-1}\bar{x}_{k-1}$ since $\bar{x}_k \in \text{span}\{\bar{x}_1, \ldots, \bar{x}_{k-1}\}$.

F **1.e.** Let $A \in \mathbb{R}^{m\times n}$. Then $\dim(\text{Null}(A)) = \dim(\text{Null}(A^T))$.

only true if $m = n$:

$\dim(\text{row}(A)) + \dim(\text{null}(A)) = n$

but $\dim(\text{row}(A^T)) + \dim(\text{null}(A^T)) = m$.

2. The matrix A is nonsingular. Compute its inverse *without* using determinants or co-factors.

$$A = \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & -5 & -2 \\ 0 & 1 & -3 & -1 \end{array}\right)$$

$R_2 + 3R_1 \to R_2$

$-R_2 \to R_2$, then

$R_1 + 2R_2 \to R_1$

so, $\boxed{A^{-1} = \begin{pmatrix} -5 & -2 \\ -3 & -1 \end{pmatrix}}$

3. Find a basis for the null space of the matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & -2 & -1 & -2 \\ -1 & 3 & 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & -2 & -1 & -2 & 0 \\ -1 & 3 & 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 4 & 3 & 0 & 0 \\ 0 & 4 & 3 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$2R_1 - R_2 \to R_2$
$R_1 + R_3 \to R_3$

$R_2 - R_3 \to R_3$, then
$\frac{1}{4}R_2 \to R_2$

then,
$x_1 = -x_2 - x_3 + x_4 = \frac{3}{4}\alpha - \alpha + \beta = -\frac{1}{4}\alpha + \beta$
$x_2 = -\frac{3}{4}x_3 = -\frac{3}{4}\alpha$
$x_3 = \alpha$
$x_4 = \beta$

$$\text{Null}(A) = \alpha \begin{pmatrix} -1 \\ -3 \\ 4 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

or $\text{Null}(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ -3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

4. Consider the matrix $A \in \mathbb{R}^{3\times 3}$:

$$A = \begin{pmatrix} 4 & 2 & -1 \\ -2 & 0 & 0 \\ 8 & -2 & 1 \end{pmatrix}$$

a.) Find the *LU* factorization of *A*.

b.) Use your answer to part *a.*) to compute the determinant of *A*.

a.) $R_2 - \boxed{-\tfrac{1}{2}} R_1 \to R_2$ $\quad \begin{pmatrix} 4 & 2 & -1 \\ 0 & 1 & -\tfrac{1}{2} \\ 0 & -6 & 3 \end{pmatrix} \to \begin{pmatrix} 4 & 2 & -1 \\ 0 & 1 & -\tfrac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$

$R_3 - \boxed{2} R_1 \to R_3$

$R_3 - \boxed{-6} R_2 \to R_3$

Thus,

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\tfrac{1}{2} & 1 & 0 \\ 2 & -6 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 4 & 2 & -1 \\ 0 & 1 & -\tfrac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}}_{U}$$

b.) $\det(A) = \det(L) \cdot \det(U) = (1 \cdot 1 \cdot 1) \cdot (4 \cdot 1 \cdot 0) = 1 \cdot 0 = 0$.

5. Solve the linear system of equations using Cramer's Rule. Give your answers as reduced fractions. You must use Cramer's Rule to receive credit.

$$\begin{cases} 3x_1 + 5x_2 &= -2 \\ 2x_1 + 4x_2 &= \ \ 1 \end{cases}$$

$A = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$ $\quad |A| = 12 - 10 = 2$

$A_1 = \begin{pmatrix} -2 & 5 \\ 1 & 4 \end{pmatrix}$ $\quad |A_1| = -8 - 5 = -13$

$A_2 = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}$ $\quad |A_2| = 3 + 4 = 7$

so, $x_1 = \frac{-13}{2}$ and $x_2 = \frac{7}{2}$

the solution is

$$\bar{x} = \begin{pmatrix} -13/2 \\ 7/2 \end{pmatrix}$$

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M511: Linear Algebra

Summer 2018

Midterm Exam: Chapters 1–3 (part II)

WICHITA STATE UNIVERSITY

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6. Consider the matrix

$$A = \underset{E_5}{\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}} \underset{E_4}{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}} \underset{E_3}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \underset{E_2}{\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}} \underset{E_1}{\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}}.$$

Compute the determinant of A.

$$\det(A) = \det(E_5) \cdots \det(E_1)$$

$$= 4 \cdot 1 \cdot (-1) \cdot \left(-\tfrac{1}{2}\right) \cdot 1 = 2$$

7. Find the values of λ for which the matrix $A \in \mathbb{R}^{3\times 3}$ is singular.

$$A = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$|A| = (1-\lambda)^2 - 4 = 0$

$(1-\lambda)^2 = 4$

$1-\lambda = \pm 2$

$\lambda - 1 = \pm 2$

$\lambda = 1 \pm 2$

the values of λ that make A singular are

$\boxed{\lambda = -1, 3}$

8. Solve the linear system of equations, if possible, using your favorite method.

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 - x_2 + x_3 = 3 \\ -x_1 + 2x_2 + 3x_3 = 7 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 3 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 4 & 2 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{11}{2} & 11 \end{array}\right)$$

$R_2 - 2R_1 \to R_2$
$R_3 + R_1 \to R_1$

$\frac{1}{4} R_2 \leftrightarrow R_3$, then
$R_3 + 5R_2 \to R_3$

$\frac{2}{11} R_3 \to R_3$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$x_1 = 1 - 2x_2 + x_3 = 1 - 2 + 2 = 1$

$x_2 = 2 - \frac{1}{2} x_3 = 2 - 1 = 1$

$x_3 = 2$

Solution is

$\hat{x} = (1, 1, 2)^T$

9. Consider the subset of $C^\infty(\mathbb{R})$, $S = \text{span}\{e^t, e^{-t}\}$.

a.) What is the dimension of S? Justify your answer.

b.) Recall that $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$. Find the transition matrix from $\{e^t, e^{-t}\}$ to $\{\cosh(t), \sinh(t)\}$.

a.) $W(e^t, e^{-t}) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -1-1 = -2 \neq 0.$

Since e^t, e^{-t} are linearly independent, then $\dim(S) = 2$.

b.) $H = \{\cosh(t), \sinh(t)\}$ $\quad E = \{e^t, e^{-t}\}$

$[\cosh t] = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, $\quad [\sinh t] = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

$H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad \det H = -1/2$

$H^{-1} = -2\left(\frac{1}{2}\right)\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

so the transition matrix from $\{e^t, e^{-t}\}$ to $\{\cosh t, \sinh t\}$

is $\boxed{H^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$

10. Consider the following vectors in $\mathbb{R}^2$,

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Write $\mathbf{x} = 4\mathbf{v}_1 - 2\mathbf{v}_2$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

$S: V \to U$, $S = U^{-1}V$

$U = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$, $U^{-1} = \frac{-1}{4}\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$, $V = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

$$S = -\frac{1}{4}\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \frac{-1}{4}\begin{pmatrix} 4 & -11 \\ -4 & 10 \end{pmatrix}$$

$$[\bar{x}]_u = S[\bar{x}]_v = \frac{-1}{4}\begin{pmatrix} 4 & -11 \\ -4 & 10 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \frac{-1}{4}\begin{pmatrix} 38 \\ -36 \end{pmatrix}_u$$

so $\boxed{\bar{x} = -\frac{19}{2}\bar{u}_1 + 9\bar{u}_2}$