Math 511: Linear Algebra

Su2018

Midterm Review Guide – Brief Solutions

1. $$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 3 & 7 \end{array}\right) \xrightarrow[R_3+R_1 \to R_1]{R_2-2R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 4 & 2 & 8 \end{array}\right) \xrightarrow[R_3+5R_2 \to R_3]{1/4\, R_2 \leftrightarrow R_3,\ \text{then}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 11/2 & 11 \end{array}\right)$$

$$\xrightarrow{\frac{2}{11} R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$x_1 = 1 - 2x_2 + x_3 = 1 - 2 + 2 = 1$

$x_2 = 2 - \frac{1}{2}x_3 = 2 - 1 = 1$

$x_3 = 2$

Solution is $\hat{x} = (1, 1, 2)^T$

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2. $$\left(\begin{array}{cccc|c} 1 & 5 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = 3 - 5\alpha + 2\beta$

$x_2 = \alpha \in \mathbb{R}$

$x_3 = \beta \in \mathbb{R}$

$x_4 = 6$

$$\bar{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 6 \end{pmatrix} + \alpha \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \alpha, \beta \in \mathbb{R}.$$

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3. $$AB = \begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 6+5+4 & 3+15+1 \\ -4+0+8 & -2+0+2 \end{pmatrix} = \begin{pmatrix} 15 & 19 \\ 4 & 0 \end{pmatrix}$$

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4. $$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$RR^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \cos\theta\sin\theta - \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $R^T = R^{-1}$.

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5. $$A = \begin{pmatrix} -2 & 4 \\ 6 & 8 \end{pmatrix} \xrightarrow{1.\ R_2+3R_1 \to R_2} \begin{pmatrix} -2 & 4 \\ 0 & 20 \end{pmatrix} \xrightarrow[3.\ -\frac{1}{2} R_1 \to R_1]{2.\ \frac{1}{20} R_2 \to R_2} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1+2R_2 \to R_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, $$A^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/20 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

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6. $$\left(\begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\substack{R_2-2R_1 \to R_2 \\ R_3-2R_1 \to R_3}]{R_1 \leftrightarrow R_2,\ \text{then}} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 \end{array}\right) \xrightarrow[R_3+R_2 \to R_3]{R_2 \leftrightarrow -R_3,\ \text{then}} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_1-3R_2 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -5 & 3 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

So, $$\boxed{A^{-1} = \begin{pmatrix} 0 & -5 & 3 \\ 0 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}}$$

7. $\begin{pmatrix} A^{-1} \\ I \end{pmatrix} \begin{pmatrix} A & I \end{pmatrix} = \begin{pmatrix} A^{-1}A & A^{-1} \\ A & I \end{pmatrix} = \begin{pmatrix} I & A^{-1} \\ A & I \end{pmatrix}$

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8. $A = \begin{pmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{pmatrix} \to \begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} = U$

$R_2 - \boxed{-2}\, R_1 \to R_2$
$R_3 - \boxed{3}\, R_1 \to R_3$
$R_3 - \boxed{-2}\, R_2 \to R_3$

$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}$ So, $\boxed{A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}}$

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9. a.) $|A| = 2\begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix} - 3\begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} + 5\begin{vmatrix} -2 & 4 \\ 1 & -1 \end{vmatrix} = 2(12+2) - 3(-6-2) + 5(2-4) = 28 + 24 - 10 = 42.$

b.) Inverse via cofactor method:

$M_{11} = \begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$ $M_{12} = \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix}$ $M_{13} = \begin{pmatrix} -2 & 4 \\ 1 & -1 \end{pmatrix}$

$M_{21} = \begin{pmatrix} 3 & 5 \\ -1 & 3 \end{pmatrix}$ $M_{22} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ $M_{23} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

$M_{31} = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$ $M_{32} = \begin{pmatrix} 2 & 5 \\ -2 & 2 \end{pmatrix}$ $M_{33} = \begin{pmatrix} 2 & 3 \\ -2 & 4 \end{pmatrix}$

$C_A = \begin{pmatrix} 14 & 8 & -2 \\ -14 & 1 & 5 \\ -14 & -14 & 14 \end{pmatrix}$, so $\boxed{A^{-1} = \frac{1}{42}\begin{pmatrix} 14 & -14 & -14 \\ 8 & 1 & -14 \\ -2 & 5 & 14 \end{pmatrix}}$

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10. $A = \begin{pmatrix} 5 & 7 \\ -8 & 3 \end{pmatrix}$ $|A| = 15 + 56 = 71$

$A_1 = \begin{pmatrix} 1 & 7 \\ -1 & 3 \end{pmatrix}$ $|A_1| = 3 + 7 = 10$ so $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10/71 \\ 3/71 \end{pmatrix}$

$A_2 = \begin{pmatrix} 5 & 1 \\ -8 & -1 \end{pmatrix}$ $|A_2| = -5 + 8 = 3$

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11. $1 = \det(I) = \det(Q^{-1}Q) = \det(Q^TQ) = \det(Q^T)\cdot\det(Q) = \det Q \cdot \det Q = (\det Q)^2$ so $\det Q = \pm 1$

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12. $A = \begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix}$, $|A| = (1-\lambda)(-1-\lambda) - \sqrt{3}^2 = \lambda^2 - 1 - 3 = \lambda^2 - 4 = (\lambda-2)(\lambda+2)$

So A is singular iff $\lambda = \pm 2$.

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13. $\det(A) = \det(LU) = \det(L)\det(U) = \det U = 2\cdot 1\cdot(-2)\cdot 5 = -20.$

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14. $\bar{x} = 4\bar{u}_1 - 2\bar{u}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}_u$. The transition matrix from U to V is $S = V^{-1}U$.

$U = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$, $V^{-1} = \frac{1}{1}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

So, $S = V^{-1}U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 11 \\ 4 & 4 \end{pmatrix}$ and $[\bar{x}]_V = S[\bar{x}]_U = \begin{pmatrix} 10 & 11 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix}_u = \begin{pmatrix} 18 \\ 8 \end{pmatrix}_V$

$\boxed{\bar{x} = 18\bar{v}_1 + 8\bar{v}_2}$

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15. $A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ 6 & -3 & -5 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & -15 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 5/7 \end{pmatrix}$

$R_1 \leftrightarrow R_2$, then
$R_2 + 3R_1 \to R_2$
$R_3 - 6R_1 \to R_3$

$\frac{1}{7}R_2 \to R_2$, then
$R_3 + 15R_2 \to R_3$

$x_1 = -4/7\,t - 5/7\,t + 2t = 5/7\,t$
$x_2 = 2/7\,t$
$x_3 = -5/7\,t$
$x_4 = t$

So, $\text{Row}(A) = \text{span}\{(1,2,-1,-2), (0,1,0,-2/7), (0,0,1,5/7)\}$

$\text{Col}(A) = \text{span}\left\{\begin{pmatrix} -3 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}\right\} = \mathbb{R}^3$ so any basis is acceptable.

$\text{Null}(A) = \text{span}\{(5,2,-5,7)^T\}$

16. a) RREF: $\begin{pmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ b.) $Col(A) = span\{\bar{a}_1, \bar{a}_2, \bar{a}_5\} = \mathbb{R}^3$.

c.) $x_5 = 0, x_4 = \beta, x_3 = \alpha, x_2 = \alpha + \beta, x_1 = -\alpha - 3\beta \Rightarrow Null(A) = span\{(-1,1,1,0,0),(-3,1,0,1,0)\}$.

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17. All 10 axioms must be checked: $\bar{x} = x, \bar{y} = y, \bar{z} = z$ in $(\mathbb{R}, \oplus, \otimes)$ and α, β in $(\mathbb{R}, +, \cdot)$

C1. $\alpha \otimes \bar{x} = \alpha x + (1-\alpha)b \in \mathbb{R}$ ✓

C2. $\bar{x} \oplus \bar{y} = x + y - b \in \mathbb{R}$ ✓

A1. $\bar{x} \oplus \bar{y} = x + y - b = y + x - b = \bar{y} \oplus \bar{x}$ ✓

A2. $\bar{x} \oplus (\bar{y} \oplus \bar{z}) = x + (y + z - b) - b = x + y + z - b - b = (x + y - b) + z - b = (\bar{x} \oplus \bar{y}) \oplus \bar{z}$ ✓

A3. $\bar{x} = \bar{x} + \bar{0} = x + ? - b = x$ implies $? = b$,
So the zero vector in this space is $\boxed{\bar{0} = b}$ ✓

A4. $\bar{x} \oplus (-\bar{x}) = \bar{0} \Rightarrow x + (-\bar{x}) - b = b \Rightarrow \boxed{-\bar{x} = -x + 2b}$ This exists for each $\bar{x} \in (\mathbb{R}, \oplus, \otimes)$ ✓

A5. $\alpha \otimes (\bar{x} \oplus \bar{y}) = \alpha(\bar{x} \oplus \bar{y}) + (1-\alpha)b = \alpha(x + y - b) + (1-\alpha)b = \alpha x + \alpha y - \alpha b + b - \alpha b$

$= \alpha x + b - \alpha b + \alpha y + \underline{b} - \alpha b \, \underline{-b} = \alpha x + (1-\alpha)b + \alpha y + (1-\alpha)b - b$

$= (\alpha \otimes \bar{x}) \oplus (\alpha \otimes \bar{y})$ ✓

A6. $(\alpha + \beta) \otimes \bar{x} = (\alpha + \beta)x + (1 - (\alpha + \beta))b = \alpha x + \beta x + b - \alpha b - \beta b$

$= \alpha x + b - \alpha b + \beta x + \underline{b} - \beta b \, \underline{-b} = \alpha x + (1-\alpha)b + \beta x + (1-\beta)b - b$

$= (\alpha \otimes \bar{x}) \oplus (\beta \otimes \bar{x})$ ✓

A7. $(\alpha\beta) \otimes \bar{x} = \alpha\beta x + (1 - \alpha\beta)b = \alpha(\beta x) + b - \alpha(\beta b) = \alpha(\beta x + \underline{b} - \beta b) \, \underline{-\alpha b} + b = \alpha(\beta x + (1-\beta)b) + (1-\alpha)b$

$= \alpha \otimes (\beta \otimes \bar{x})$

One can (and you should) show in a similar fashion that this is equivalent to $\beta \otimes (\alpha \otimes \bar{x})$.

A8. $1 \otimes \bar{x} = 1x + (1-1)b = x = \bar{x}$ ✓