Name: Key

M511: Linear Algebra (Summer 2018)

Good Problems 2: Chapter 2

Instructions *Complete all problems on this paper, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Let $A, B \in \mathbb{R}^{3\times 3}$ with $\det(A) = 4$ and $\det(B) = 6$, and let E be an elementary matrix of type I. Determine the value of each of the following:

a.) $\det\left(\frac{1}{2}A\right) = \left(\frac{1}{2}\right)^3 \det(A) = \frac{1}{8} \cdot 4 = \frac{1}{2}$

b.) $\det(B^{-1}A^T) = \det(B^{-1}) \cdot \det(A^T) = \frac{1}{\det(B)} \cdot \det(A) = \frac{4}{6} = \frac{2}{3}$

c.) $\det(EA^2) = \det(E) \cdot \det(A)^2 = -1 \cdot 4^2 = -16$

2. If $A \in \mathbb{R}^{n\times n}$ is nonsingular, show that $A^T A$ is nonsingular and $\det(A^T A) > 0$.

$\det(A^T A) = \det(A^T) \cdot \det(A) = \det(A)^2 \geq 0.$

Since A is nonsingular, $\det(A) \neq 0$. Thus $\det(A^T A) = \det(A)^2 \neq 0$.

Moreover, $\det(A^T A) = \det(A)^2 > 0$.

3. Let $A \in \mathbb{R}^{n\times n}$ and let λ be a scalar. Show that $\det(A - \lambda I) = 0$ if and only if $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$.

$\det(A-\lambda I)=0$ if and only if $(A-\lambda I)$ is singular.

By Thm 1.5.2, $(A-\lambda I)$ is singular if and only if there exists $\bar{x}\neq\bar{0}$ satisfying

$(A-\lambda I)\bar{x}=\bar{0}$.

By matrix algebra rules,

$$(A-\lambda I)\bar{x}=\bar{0} \Rightarrow A\bar{x}-\lambda I\bar{x}=\bar{0}$$
$$\Rightarrow A\bar{x}-\lambda\bar{x}=\bar{0}$$
$$\Rightarrow A\bar{x}=\lambda\bar{x} \quad \blacksquare$$

4. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{y}$, and let $A \in \mathbb{R}^{n\times n}$. Show that if $A\mathbf{x} = A\mathbf{y}$, then $\det(A) = 0$.

$$A\bar{x}=A\bar{y} \Rightarrow A\bar{x}-A\bar{y}=\bar{0}$$
$$\Rightarrow A(\bar{x}-\bar{y})=\bar{0}$$

Since $\bar{x}\neq\bar{y}$, then $\bar{x}-\bar{y}\neq\bar{0}$, hence $(\bar{x}-\bar{y})$ is a nontrivial solution to the homogeneous equation. By theorem 1.5.2, A is singular.

Therefore $\det(A)=0$.

5. Let

$$A = \begin{pmatrix} x & 1 & 1 \\ 1 & x & -1 \\ -1 & -1 & x \end{pmatrix}.$$

a.) Compute all minors and cofactors of A.

b.) Compute $\det(A)$. (Your answer should be a function of x.)

c.) For what values of x will the matrix be singular?

a) $M_{11} = \begin{pmatrix} x & -1 \\ -1 & x \end{pmatrix}$ $\quad M_{12} = \begin{pmatrix} 1 & -1 \\ -1 & x \end{pmatrix}$ $\quad M_{13} = \begin{pmatrix} 1 & x \\ -1 & -1 \end{pmatrix}$

$M_{21} = \begin{pmatrix} 1 & 1 \\ -1 & x \end{pmatrix}$ $\quad M_{22} = \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$ $\quad M_{23} = \begin{pmatrix} x & 1 \\ -1 & -1 \end{pmatrix}$

$M_{31} = \begin{pmatrix} 1 & 1 \\ x & -1 \end{pmatrix}$ $\quad M_{32} = \begin{pmatrix} x & 1 \\ 1 & -1 \end{pmatrix}$ $\quad M_{33} = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$

The cofactor matrix is,

$$C_A = \begin{pmatrix} x^2-1 & 1-x & x-1 \\ -x-1 & x^2+1 & x-1 \\ -x-1 & x-1 & x^2-1 \end{pmatrix}$$

b) $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = x(x^2-1) + 1\cancel{(1-x)} + 1\cancel{(x-1)}$

$= \boxed{x(x^2-1)}$

c) A will be singular iff $\det(A) = 0$.

$x(x^2-1) = 0$

$x(x+1)(x-1) = 0$

$\boxed{x = 0, -1, 1}$

6. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

a.) Compute the *LU* factorization of *A*.

b.) Use the *LU* factorization to determine the value of $\det(A)$.

a) $\left[\begin{array}{l} R_2 - \boxed{1} R_1 \to R_2 \\ R_3 - \boxed{1} R_1 \to R_3 \\ R_4 - \boxed{1} R_1 \to R_4 \end{array}\right.$ $\left[\begin{array}{l} R_3 - \boxed{2} R_2 \to R_3 \\ R_4 - \boxed{3} R_2 \to R_4 \end{array}\right.$ $\left[\, R_4 - \boxed{3} R_3 \to R_4 \right.$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So $$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b) $\det(A) = \det(L) \cdot \det(U) = \det(U) = 1.$