Def'n. A Taylor Expansion T_a at a point $a \in \mathbb{R}^n$ is an arbitrary series

$$T_a = \sum_{\alpha} \frac{\lambda_{\alpha,a}}{\alpha!}(x-a)^\alpha \qquad \lambda_{\alpha,a} \in \mathbb{R}$$

for a family $\{T_a\}_{a \in U}$, $U \subset \mathbb{R}^n$,

$$\lambda_\alpha : U \to \mathbb{R} \qquad a \longmapsto \lambda_{\alpha,a}$$

Whitney's Extension Theorem
Let $K \subset \mathbb{R}^n$ be compact and $\{T_a\}_{a \in K}$ a family of Taylor Expansions on K. There exists $f \in C^\infty(\mathbb{R}^n)$ such that $T_a f = T_a$ if and only if for any multiindex α, $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall\, x,y \in K$ such that $\|x-y\| < \delta$, then

$$(*) \quad \left| \lambda_{\alpha,y} - \sum_{|\beta|=0}^{m} \frac{\lambda_{\alpha+\beta,x}}{\beta!}(y-x)^\beta \right| < \varepsilon \|y-x\|^m$$

Lemm 1A. Let $a_k \to 0$ be a sequence in $\mathbb{R}^n$. Assume $\exists$ $c \in \mathbb{N}$ such that

$$\|a_k\|, \|a_\ell\| < c\,\|a_k - a_\ell\| \qquad \forall\, k \neq \ell$$

and let $\{T_{a_k}\}_{k \in \mathbb{N}}$ be a family of Taylor Expansions w/ $T_0 = 0$.
If $(*)$ holds at the origin, then there exists a global smooth function $f \in C^\infty(\mathbb{R}^n)$ s.t.

$$T_{a_k} f = T_{a_k} \quad \text{and} \quad T_0 f = 0.$$

Proof: [N. S] Lemma 7.4 ■

Let $\mathcal{E}_1, \mathcal{E}_2 \to M$ be sheaves of vector spaces over M.

In reality, $E_1, E_2 \to M$ are vector bundles over M and $\mathcal{E}_1, \mathcal{E}_2$ are the sheaves of germs of smooth sections of E_1 and E_2.

The stalks of $\mathcal{E}_1$ and $\mathcal{E}_2$ are vector spaces.

A linear morphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ is a smooth map of the ringed spaces $\mathcal{E}_1, \mathcal{E}_2$ that are "stalk-preserving" and linear along stalks.

<u>Lemma 1B.</u> Let $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ be a linear morphism of sheaves. For any $p \in M$, there exists an open nbhd $p \in U \subset M$ and a natural number $k \in \mathbb{N}$ such that

For all $q \in U$, $j^k s_1(q) = j^k s_2(q)$ implies

$$\varphi(s_1)(q) = \varphi(s_2)(q).$$

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "\varphi"] \arrow[dr] & \mathcal{E}_2 \arrow[d] \\
& M \arrow[ul, bend left, "s_1"] \arrow[ul, bend right, "s_2"'] \arrow[u, bend right]
\end{tikzcd}$$

$\varphi(s_1) = \varphi(s_2)$ on U.

Proof: We can assume $M = \mathbb{R}^n$ and let $E_1 = \mathbb{R}^n \times \mathbb{R}$, the trivial line bundle, and $p = \underline{0} \in \mathbb{R}^n$.

RE. Justify the reduction above.

Then we need to prove that $\exists$ $\underline{0} \in U \subset \mathbb{R}^n$ and $k \in \mathbb{N}$ such that

$$j^k_y f = 0 \quad \text{implies} \quad \varphi(f)(y) = 0$$

for all $y \in U$.

First prove. $\exists$ $N > 0$, an open nbhd U of $\underline{0}$ and $k \in \mathbb{N}$, such that

$$j^k_y f = 0 \Rightarrow |\varphi(f)(y)| < N$$

for all $y \in U \setminus \{0\}$.

If not, then considering the nbhds $U_m := \{ \|x\| \leq \frac{1}{m} \}$ we can produce a sequence of pts $x_m \to 0$, $x_m \in U_m \setminus \{0\}$ and we can find h_m such that

$$j^k_{x_m} h_m = 0 \quad \text{but} \quad |\varphi(h_m)(x_m)| > m$$

By Lemma 1A, $\exists$ a smooth global function $f \in \mathcal{E}(\mathbb{R}^n)$ such that

$$j^\infty_{x_k} f = j^\infty_{x_k} h_k, \quad j^\infty_0 f = 0.$$

(cont'd on next page)

$U \setminus \{0\}$ ← set minus notation

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Thursday, February 23, 2017 1:39 PM

This implies $\varphi(f)(x_k) = \varphi(h_k)(x_k)$, and $\varphi(f)(0) = 0$, and we obtain a contradiction:

$$\varphi(f)(x_k) \to \varphi(f)(0) = 0 \quad \text{but} \quad |\varphi(f)(x_k)| = |\varphi(h_k)(x_k)| > k.$$

This proves the Lemma in $U \setminus \{0\}$

To prove this at $\underline{0} \in U \subset \mathbb{R}^n$, let f be a function such that $j_0^k f = 0$. Consider a sequence of points $x_m \to 0$ such that

$$2\|x_m - x_\ell\| > \|x_m\|, \|x_\ell\| \quad \text{for } m < \ell$$

Also consider a sequence of jets

$$\left(j_{x_m}^k f\ ?\right) = T_{x_m} = \sum_\alpha \frac{\lambda_{\alpha, x_m}}{\alpha!}(x - x_m)^\alpha \quad \text{defined by}$$

$$\lambda_{\alpha, x_m} := \begin{cases} (D_\alpha f)(x_m) & \text{if } |\alpha| \leq k \\ 0 & \text{otherwise} \end{cases}$$

By Lemm 1A, $\exists$ a global smooth function $u \in \mathcal{E}(\mathbb{R}^n)$ such that

$$j_{x_m}^\infty u = T_{x_m}, \quad j_0^\infty u = 0.$$

Then $j_{x_m}^k u = j_{x_m}^k f$ implies that $\varphi(u)(x_m) = \varphi(f)(x_m)$.

On the other hand, $\varphi(u)(0) = 0$ by linearity. So $\varphi(f)(0) = 0$. ∎

Let E_1, E_2 be vector bundles over a smooth manifold M.

The jet spaces $J^k E \to M$ are vector bundles, $J^\infty E = JE$ inherits an $\mathbb{R}$-linear structure along stalks.

Def'n. A <u>linear differential operator</u> $E_1 \rightsquigarrow E_2$ is an $\mathbb{R}$-linear morphism of ringed spaces over M,

$$P : J^\infty E_1 \to E_2 .$$

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the sheaves of smooth sections, given a Linear Differential Operator $P : J^\infty E_1 \to E_2$, we can define a linear morphism of sheaves by

lives in sections of $\mathcal{E}_2$.

$$\varphi_P : \mathcal{E}_1 \to \mathcal{E}_2 : s_1 \longmapsto P(j^\infty s_1)$$

<u>Peetre's Theorem</u>. The map $P \longmapsto \varphi_P$ is a linear isomorphism

$$\mathrm{Diff}_{\mathbb{R}}(E_1, E_2) \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{E}_1, \mathcal{E}_2)$$

where $\mathrm{Diff}_{\mathbb{R}}(E_1, E_2)$ is the collection of all linear diff. operators.

<u>Proof</u>: Let $\varphi : \mathcal{E}_1|_V \to \mathcal{E}_2|_V$ be a $\mathbb{R}$-linear sheaf morphism. Fix a point $x \in V$. We need to verify that on a sufficiently small nbhd $U \ni x$, $\varphi = \varphi_P$ is determined by an L.D.O. of finite order.

By Lemma 1B, $\exists$ a nbhd $U \subset V$ of x and $k \in \mathbb{N}$ such that the map $P : (J^k E_1)|_U \to E_2|_U$ (cont'd on next)

$$P : j^k_y s \longmapsto \varphi(s)(y) \quad \text{for } y \in U \text{ is well-defined.}$$

P is $\mathbb{R}$-linear, we need to show it is smooth, WLOG E_1, E_2 are trivial line bundles over $\mathbb{R}^n$. Therefore locally

$$Pf = \sum_{|\alpha|=0}^{k} a_\alpha D_\alpha f(p)$$

R.E. The smoothness of the a_α is "easily achieved" by induction using $P(j^k f) = \varphi(f)$ is smooth for any k-jet $j^k f$ of f. ∎

Locally

Defn. A (linear) differential operator is given locally by

$$Pf = \sum_{|\alpha| \le k} a_\alpha D_\alpha f \quad \text{where } a_\alpha \in \mathcal{F}(M).$$

(Local) Peetre Theorem. Let $E_1 \to M$ and $E_2 \to M$ be vector bundles over the same connected base manifold and suppose $P : \Gamma E_1 \to \Gamma E_2$ is a linear morphism of sheaves satisfying $\operatorname{supp}(P\sigma) \subseteq \operatorname{supp}(\sigma)$ for all $\sigma \in \Gamma E_1$. Then P is a linear diff. operator.

Go through Parkers Proof for this.