Slovák's Theorem

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Def'n: A morphism of sheaves $\varphi : F \to F'$ is regular if, for any smooth family of sections $\{s_t : U_t \to F\}_{t \in T}$, the family $\{\varphi(s_t) : U_t \to F'\}_{t \in T}$ is also smooth.

$$\begin{tikzcd} F \arrow[rr, "\varphi"] \arrow[dr, shift right] & & F' \arrow[dl, shift left] \\ & M \arrow[ul, shift right, "s_t"'] \arrow[ur, shift left, "\varphi(s_t)"'] & \end{tikzcd}$$

Let $F \to M$, $\bar{F} \to M$ be fiber bundles over M.
Let $\mathcal{F}$ and $\bar{\mathcal{F}}$ be the sheaves of smooth sections of F, $\bar{F}$.

Def'n: A differential operator $F \rightsquigarrow \bar{F}$ is a morphism of ringed spaces over M:

$$P : J^\infty F \to \bar{F}.$$

Any diff. op. P defines a morphism of sheaves

$$\varphi_P : \mathcal{F} \to \bar{\mathcal{F}} : s(x) \longmapsto P(j^\infty_x s)$$

each representation of x in $\mathcal{F}$ maps to a single element in $\bar{\mathcal{F}}$.

This map φ_P is regular, since

$$\varphi_P(s_t) : (t, x) \longmapsto (t, P(j^\infty_x s_t))$$ is smooth in t.

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Theorem (Slovák). Let $\mathcal{F}$, $\bar{\mathcal{F}}$ be sheaves of smooth sections of smooth fiber bundles $F, \bar{F}$ over M.
The map $P \mapsto \varphi_P$ is a bijection

$$\mathrm{Hom}_{reg}(\mathcal{F}, \bar{\mathcal{F}}) \cong \mathrm{Diff}(F, \bar{F})$$

(Regular morphism perserves an idea of "smoothness.")

Proof: Let $\varphi : \mathcal{F} \to \bar{\mathcal{F}}$ be a regular morphism of sheaves.

Lemma. If $\varphi : \mathcal{F} \to \bar{\mathcal{F}}$ is a regular morphism of sheaves, then the map $P_\varphi : J^\infty F \to \bar{F}$ Locally factors through some finite jet space.

$\forall\ j^\infty_x s \in J^\infty F$, $\exists\ V \ni j^\infty_x s$ and $\exists\ k \in \mathbb{N}$ s.t.

$$\begin{tikzcd} V \arrow[rr, "P_\varphi"] \arrow[dr, "\pi_k"'] & & \bar{F} \\ & J^k F \arrow[ur, "\exists!"'] & \end{tikzcd}$$

"Proof": Use Whitney Extension Theorem. See [NS]. ∎

Remains to Prove: $P_k : J^k F \to \bar{F} : j^k_x s \mapsto \varphi(s)(x)$ is smooth.

W.L.O.G: $M = \mathbb{R}^n$ and $F = \mathbb{R}^n \times \mathbb{R}^r$ (base $\times$ fiber).
(more accurate: $U \subset \mathbb{R}^n$ $\quad U \times V \subset \mathbb{R}^n \times \mathbb{R}^r$, U, V open)

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Let ξ be the <u>universal</u> family of k-jets. The smoothness of P_k follows from the regularity of φ by

$$P_k(j^k_x s) = \varphi(\xi_f)(x)$$

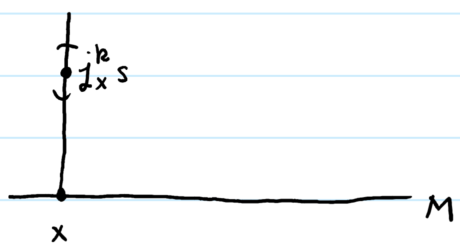
where f is the only polynomial whose k-jet is $j^k_x s$. ▨

Ex. The <u>universal family</u> ξ is defined in

$$\mathcal{U} \subseteq J^k F \overset{loc}{=} P_0|_k(\mathbb{R}^n, \mathbb{R}^r) \times \mathbb{R}^n.$$

$$\xi : \mathcal{U} \to F = \mathbb{R}^n \times \mathbb{R}^r : (f, x) \longmapsto (f(x), x)$$

locally, $\xi_f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^r$ is defined by $\xi_f = [f]_x$, $x \in \mathcal{U}$.



ξ assigns to f its related germ.

The point? ξ is a universal object.

A picture

$$\begin{tikzcd}
F \arrow[rr, squiggly, "P"] \arrow[dr, shift left] & & \bar{F} \arrow[dl, shift left] \\
& M \arrow[ul, shift left, "\sigma"] \arrow[ur, shift left, "P(\sigma)"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F} \arrow[rr, "\varphi_P"] \arrow[dr, shift left] & & \bar{\mathcal{F}} \arrow[dl, shift left] \\
& M \arrow[ul, shift left, "\sigma"] \arrow[ur, shift left, "P(j^\infty \sigma)"'] &
\end{tikzcd}$$

So, $\exists$ $P j^k \sigma = P j^\infty \sigma$. So the morphisms have to retain regularity when we drop linearity.

An example

Let $\pi : E \to M$ be a smooth fiber bundle w/ model fiber F.

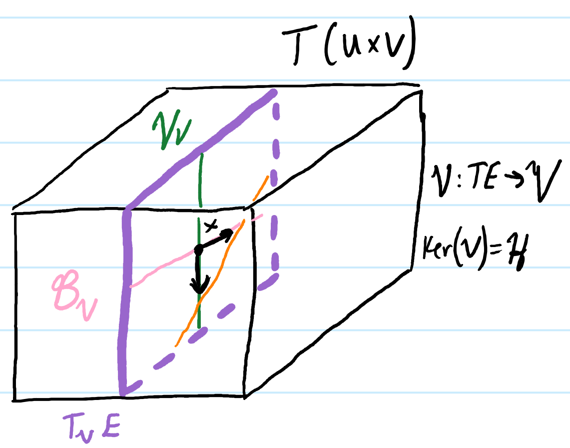
<u>Recall</u>:

$$\begin{tikzcd} TE \arrow[r, "\pi_*"] \arrow[d, "\pi_T"'] & TM \arrow[d, "\pi_T"] \\ E^{n+k} \arrow[r, "\pi"'] & M^n \end{tikzcd}$$

π is a submersion, so π_* is surjective on each fiber and $\dim(\operatorname{Ker}(\pi_*)) = k$

The <u>vertical bundle</u> is $\mathcal{V}E \to E$ a subbundle of TE whose fibers are $\mathcal{V}_v E := \operatorname{Ker}_v \pi_*$.

The elements of $\mathcal{V}_v$ are vectors that are tangent to the fibers of E. (Tangent to fibers only)



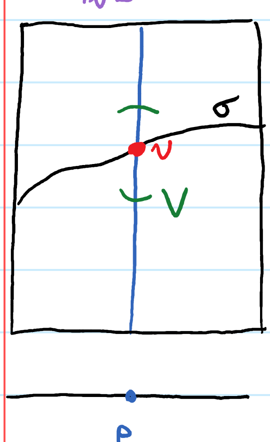
$\mathcal{B}_V$ = Basal space

In general $\mathcal{B}$ is <u>not</u> a global bundle.

(pre-connection) is a subbundle $\mathcal{H}$ of TE

<u>Def'n</u>. A horizontal distribution satisfying $TE = \mathcal{V} \oplus \mathcal{H}$

$\mathcal{H}$ is made up of vectors linearly coord of $\mathcal{V}$.



$E|_u \cong u \times F$

$\downarrow$

$E|_{u \times V} \cong u \times V$

$u \subset M$

we can define a projection

$\Gamma : \mathfrak{X}(u) \times \Gamma(E|_u) \to \Gamma(\mathcal{V}|_u)$

$\Gamma(X, \sigma)(p) = -\mathcal{V}_{\sigma(p)}(X_p)$. This is the Christoffel form of the $\mathcal{H}$. $\Gamma : E \to J^1 E$

idea: By Slovák, this corresponds to a reg. morphism of sheaves: $\varphi : \mathcal{E} \to \mathcal{E}$.